SM2H

Secondary Math 2H
Unit 10 – Probability

10.1 The Fundamental Counting Principle

Introduction to Factorial Notation

Example: There are 8 people running a race. How many different outcomes for the race are there?

Solution: There are 8 different people who can finish first. Once someone finishes first, there are only 7 people left competing for second place, then six left competing for third, and so on.

So, to calculate all the different outcomes for the race:

$8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = \mathbf{40{,}320}$

There are 40,320 different outcomes for the race.

The example above requires you to multiply a series of descending natural numbers: $8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$. This can be written as 8! (read 8 factorial).

8! means $8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$
$5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$
$9! = 9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = 362{,}880$

It is generally accepted that 0! = 1.

Now, what if you had to calculate 20!? Do you want to enter all of those numbers into your calculator? The factorial key on your calculator can be found by:

Examples: Simplify each expression to a single number or fraction.

1. $\dfrac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} =$

2. $\dfrac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot5\cdot4\cdot3\cdot2\cdot1} =$

3. $3! =$

4. $6! =$

5. $\dfrac{4!}{3!} =$

6. $\dfrac{6!}{4!} =$

7. $\dfrac{101!}{99!} =$

Fundamental Counting Principle:
If event M can occur in m ways and event N can occur in n ways, then event M followed by event N can occur in $m \cdot n$ ways.

> If you are making a sandwich and there are three types of bread, five types of meat, and four types of cheese available and you choose one type of bread, one type of meat, and one type of cheese, there are $3 \cdot 5 \cdot 4 = 60$ different sandwiches you can make.

Examples:
a) How many possible locker combinations are available if each combination consists of three numbers between 1 and 50?

b) How many license plates consisting of three numbers followed by three letters are possible if numbers and letters can be repeated?

c) There are eight true-false questions on a quiz. How many different answer combinations are possible?

d) A golf club manufacturer makes drivers with 4 different shaft lengths, 3 different lofts, 2 different grips, and 2 different club head materials. How many different combinations are possible?

e) For a college application, George must select one of five topics on which to write a short essay. He must select a different topic from the list for a longer essay. How many ways can he choose the topics for the two essays?

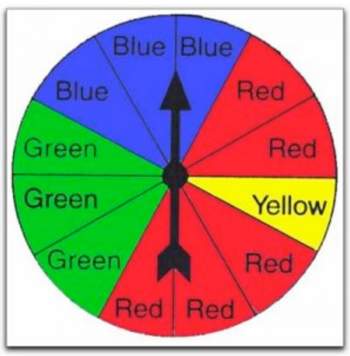
f) Abby is registering at a website. She must create a password containing 6 *different* numbers. How many possible passwords are there?

g) How many ways can 8 books be arranged on a shelf?

Probability:
A value that represents the likelihood of an event. It can be expressed as a fraction, decimal, or a percentage. A probability of 0 means that the event is impossible and a probability of 1 (or 100%) means that the event is certain to occur.

$$\text{probability} = \frac{\text{total \# of favorable outcomes in the category of interest}}{\text{total \# of possible outcomes}}$$

Example:
A spinner is pictured below. If the arrow is spun, what is the probability that the spinner lands on:



a) green

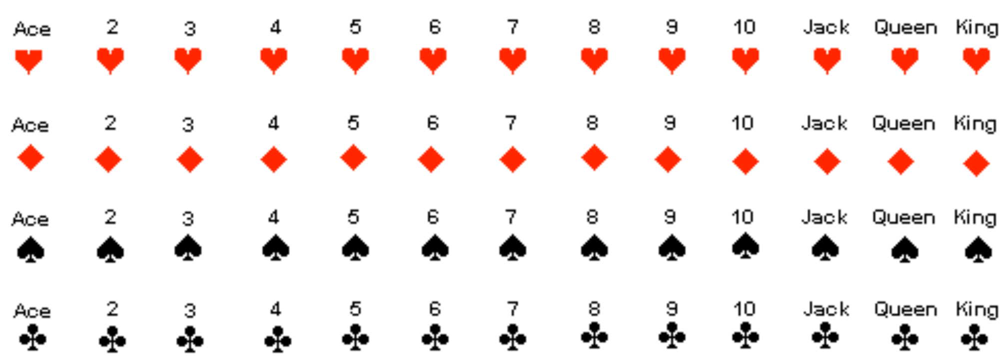
b) yellow

c) blue or red

d) not red

Example:
A card is drawn at random from a standard deck of cards. What is the probability that the card drawn is:

Sample Space for Choosing a Card from a Deck



a) a heart

b) red

c) a 5

d) red and a 4

e) not an ace

f) black or a king

10.2 Permutations and Combinations

Independent and Dependent Events

VOCABULARY -

- ***Outcome:*** The result of a single trial, experiment, or decision.
  - The possible outcomes of a coin flip would be heads or tails.
  - The possible outcomes of rolling a single die would be 1, 2, 3, 4, 5, or 6.

- ***Sample Space:*** The set of all possible outcomes.
  - The sample space for a coin flip is $\{H, T\}$.
  - The sample space for two coin flips would be $\{HH, HT, TH, TT\}$

- ***Event:*** One or more possible outcomes of a trial.
  - A coin flip coming up heads
  - Rolling either a 2 or a 6 on a die
  - Choosing a face card from a deck of cards

- ***Independent Events:*** The outcome of one event *does not* affect the outcome of another event.
  - Two coin flips are independent because the outcome of the first coin flip does not affect the outcome of the second flip.

- ***Dependent Events:*** The outcome of one event *does* affect the outcome of another event.
  - Choosing a piece of candy from a jar and then choosing a second piece without replacing the first are dependent events because taking the first piece affects what is available to be taken next.

Examples: Decide whether the events are *dependent* or *independent*.

1. Choosing an ice cream flavor and then choosing a topping for the ice cream.

2. Choosing one book on which to write an essay on and then a different book on which to give a presentation.

3. Awarding 1^{st}, 2^{nd}, and 3^{rd} prize to entries in an art contest.

4. Selecting a fiction book and a nonfiction book from the library.

Permutation and Combinations

- ***Permutation***: An arrangement of a group of objects, where order matters.
  Ex: Selecting a president, vice president, and secretary/lock combination/batting order

Permutation Formula (no repetition allowed)

${}_nP_r = P(n,r) = \dfrac{n!}{(n-r)!}$ where n is the number of things you choose from and you choose r of them

- ***Combination***: An arrangement or selection of objects in which order is not important.
  Ex: Choosing Pizza toppings/selecting people to a committee

Combination Formula (no repetition allowed)

${}_nC_r = C(n,r) = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ where n is the number of things you choose from and you choose r of them

Examples: Determine whether each situation involves a permutation or a combination.

1. In how many different orders can a person read 5 separate magazines?

2. You have just purchased 15 new songs and want to add them to a playlist. You don't want to remove any of the songs that are already on your playlist and there is only room for 5 more tracks. How many ways can you add 5 different tracks to the playlist?

3. You have 7 new movies on Netflix that you are dying to watch. You only have time to watch 3 this weekend. How many distinct ways can you watch the movies this weekend?

4. Honors English students are required to read 8 books from a list of 25. How many combinations could a student select?

5. You are Mr. Manager of a frozen banana stand. You need 5 employees and have 20 qualified applicants. How many ways can you staff the store?

6. Four members from a group of 18 on the board of directors at the Fa La La School of Arts will be selected to go to a convention (all expenses paid) in Hawaii. How many different groups of 4 are there?

Examples: Evaluate each expression

1. ${}_6P_2$

2. $P(10,5)$

3. ${}_9P_6$

4. $P(25,20)$

5. ${}_{15}C_4$

6. ${}_9C_6$

7. $C(12,4)$

8. $\binom{9}{2}$

Examples: Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities.

1. Twelve skiers are competing in the final round of the Olympic freestyle skiing aerial competition. In how many ways can 3 of the skiers finish first, second, and third to win the gold, silver, and bronze medals?

2. A pizza shop offers twelve different toppings. How many different three-topping pizzas can be formed with the twelve toppings?

3. Your English teacher has asked you to select 3 novels from a list of 10 to read as an independent project. In how many ways can you choose which books to read?

4. The school yearbook has an editor-in-chief and an assistant editor-in-chief. The staff of the yearbook has 15 students. How many different ways can students be chosen for these 2 positions?

It is important to note that when you use the permutation or combination formulas above, repetition is not allowed. In other words, you can't have the same person win first and second place.

Another Case to Consider

Example: How many different ways can the letters MATH be arranged to create four-letter "words"?

Solution: This is an example of a permutation because the order of the letters would produce a different "word" or outcome. So, we use the permutation formula:

$${}_4P_4 = P(4,4) = \frac{4!}{(4-4)!} = \frac{4!}{0!} = \frac{4\bullet 3\bullet 2\bullet 1}{1} = 24$$

But, what if some of the letters repeated? For example, how many ways can the letters in CLASSES be rearranged to create 7 letter "words"? Since the letter S repeats 3 times, some of the permutations will be the same so we will have to eliminate them. Here is how we do it. There are 7 letters to choose from and we are choosing 7 of them, so we would have the following:

$${}_7P_7 = P(7,7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7\bullet 6\bullet 5\bullet 4\bullet 3\bullet 2\bullet 1}{1} = 5040$$

Hold on – this is not our answer yet. We have to divide out our duplicate letters. As we mentioned earlier, the letter S repeats 3 times so we divide our answer by 3!:

$$\frac{5040}{3!} = 840$$

Example 7: How many ways can the letters in MISSISSIPPI be arranged to create 11-letter "words"?

Example 8:
STREETS

Example 9:
ASSIGNMENTS

10.3 Venn Diagrams

Sample Space: The set of all possible outcomes for a chance process.

Event/Subset: An outcome or set of outcomes from the sample space.

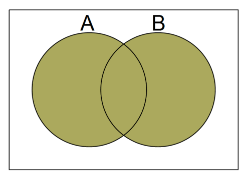
Complement (A^C)**:** "Not"

- All outcomes in the sample space that are not part of the event.

Chance Process	Sample Space	Event/Subset	Complement
Flip a coin	$S = \{\text{heads, tails}\}$	$B = \{\text{heads}\}$	$B^C = \{\text{tails}\}$
Roll a die	$S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$	even numbers $E = \{2,\ 4,\ 6\}$	$E^C = \{1,\ 3,\ 5\}$
Pick a letter in the word "probability"	$S = \{\text{P, R, O, B, A, I, L, T, Y}\}$	vowels $V = \{\text{O, A, I, Y}\}$	$V^C = \{\text{P, R, B, L, T}\}$

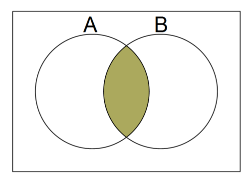
Union $(A \cup B)$**:** "Or", "Either"

- All of the elements that are in *A* or *B* or both.

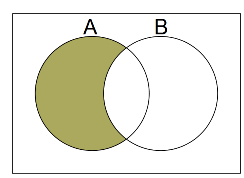


Intersection $(A \cap B)$**:** "And", "Both", "Overlap", "In common"

- All of the elements that are in *both A* and *B*.
- If the two sets don't have anything in common, the intersection is the "empty set", indicated by $\varnothing$ or $\{\ \}$.

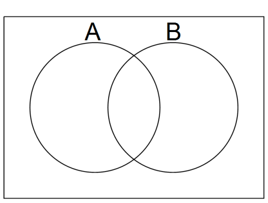


Note: If you want to write "everything in *A* that isn't in *B*," you can write either $A \cap B^C$ or $A - B$.

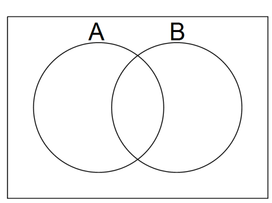


Examples: Shade the appropriate portion of the Venn diagram.

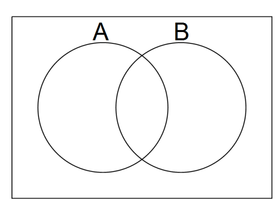
1. A^C



2. $(A \cap B)^C$



3. $B - A$



Examples:

- Chance Process: Rolling a 10-sided die.
  - Event A: Rolling an odd number
  - Event B: Rolling a prime number

a. What is the sample space?

b. List the outcomes in each event.

c. Draw a Venn diagram representing the sample space with subsets A and B.

d. List all the outcomes in $A \cup B$.

e. List all the outcomes in $A \cap B$.

f. List all the outcomes in A^C.

g. List all the outcomes in $(A \cup B)^C$.

h. List all the outcomes in $A - B$.

- Chance Process: Reaching into a messy refrigerator and grabbing a food at random.
- Sample Space: S = {broccoli, carrots, moldy cheese, milk, orange, lettuce, lime jello, bologna, egg, corn, celery}
  - Event A: Picking a vegetable
  - Event B: Picking something green

a. List the outcomes in each event.

b. Draw a Venn diagram representing the sample space with subsets A and B.

c. List all the outcomes in $A \cup B$.

d. List all the outcomes in $A \cap B$.

e. List all the outcomes in B^C.

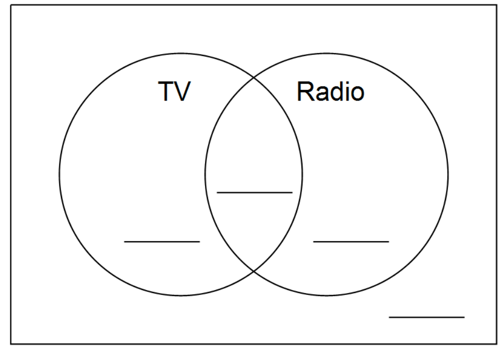
f. List all the outcomes in $(A \cap B)^C$.

g. List all the outcomes in $B - A$.

Examples:

A political ad was run on TV and on radio.

- 33% of people saw it on TV.
- 21% heard it on the radio.
- 10% of people both saw it on TV and heard it on the radio.



Determine what percent:

a) only saw it

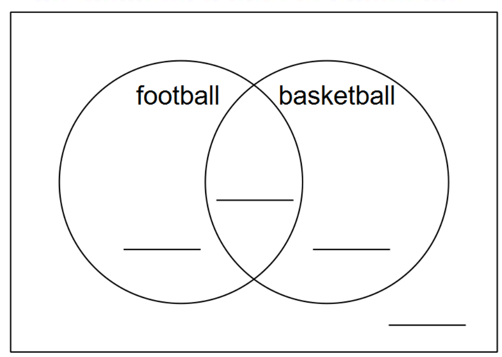
b) only heard it

c) neither heard it or saw it

d) did not see it

A sample of 60 people are asked if they enjoy watching basketball and if they enjoy watching football.

- 25 people say they enjoy watching football
- 40 people say they enjoy watching basketball
- 15 people say they enjoy watching both



Determine how many people:

a) enjoy football but not basketball

b) enjoy basketball but not football

c) don't enjoy either basketball or football

d) don't like football

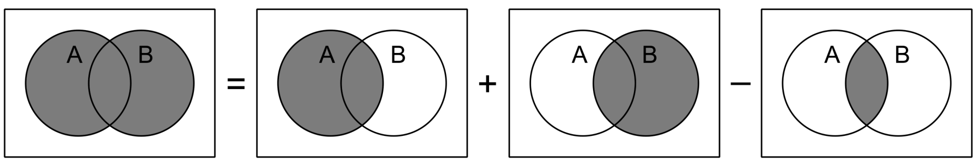
10.4 Probabilities from Venn Diagrams

Probability: A value that represents the likelihood of an event. It can be expressed as a fraction, decimal, or a percentage. A probability of 0 means that the event is impossible and a probability of 1 (or 100%) means that the event is certain to occur.

$$\text{probability} = \frac{\text{total \# of favorable outcomes in the category of interest}}{\text{total \# of possible outcomes}}$$

Remember, $(A \cap B)$ means "A and B" and $(A \cup B)$ means "A or B (or both)". With "or" probabilities, makes sure you don't count the individuals who fall in both categories twice!

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Example: In the Math Club, there are 34 students. Eleven of the students are seniors, including 7 of the 20 girls. A student is chosen at random from the club. Fill in the table and find the following probabilities:

a) $P(\text{boy})$

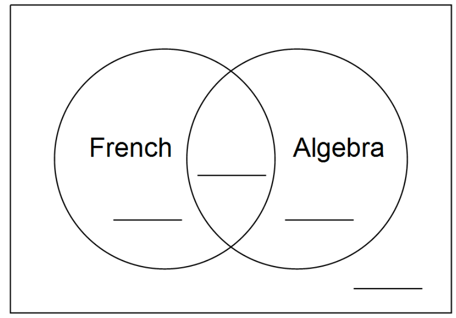
b) $P(\text{senior})$

c) $P(\text{boy} \cap \text{senior})$

d) $P(\text{girl} \cup \text{non-senior})$

	Seniors	**Non-Seniors**	**Total**
Boys			
Girls			
Total			

Example: The number of students in a high school is 1400. Of those students, 550 take French, 700 take algebra, and 400 take both French and algebra. Fill in the Venn diagram, then find the following probabilities.



a) $P(\text{does not take French})$

b) $P(\text{algebra} \cap \text{French})$

c) $P(\text{algebra, but not French})$

d) $P(\text{algebra} \cup \text{French})$

Conditional Probability: The probability of an event occurring when we already know that another event has occurred.

Example: $P(\text{lung cancer} | \text{smoke})$ would mean the probability of a person getting lung cancer given that the person smokes.

Conditional Probability Formula: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ or $\dfrac{\text{total \# in } A \cap B}{\text{total \# in } B}$

★ **"And" and "or" probabilities are fractions of the entire sample, but with conditional probabilities, the condition becomes the denominator of the fraction!**

Examples:
An ice cream shop keeps track of whether people order vanilla or chocolate ice cream and whether they order a sugar cone or a waffle cone. Fill in the table and find the requested probabilities.

	Sugar Cone	**Waffle Cone**	**Total**
Vanilla	35	26	
Chocolate	51	47	
Total			

a) $P(\text{vanilla})$

b) $P(\text{waffle})$

c) $P(\text{sugar})$

d) $P(\text{chocolate})$

e) $P(\text{vanilla} \cap \text{sugar})$

f) $P(\text{vanilla} \cap \text{waffle})$

g) $P(\text{chocolate} \cap \text{sugar})$

h) $P(\text{chocolate} \cap \text{waffle})$

i) $P(\text{vanilla} \cup \text{sugar})$

j) $P(\text{vanilla} \cup \text{waffle})$

k) $P(\text{chocolate} \cup \text{sugar})$

l) $P(\text{chocolate} \cup \text{waffle})$

m) $P(\text{vanilla} | \text{sugar})$

n) $P(\text{vanilla} | \text{waffle})$

o) $P(\text{chocolate} | \text{sugar})$

p) $P(\text{chocolate} | \text{waffle})$

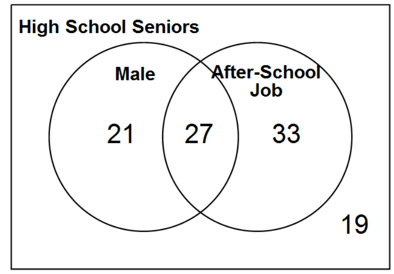
q) $P(\text{sugar} | \text{vanilla})$

r) $P(\text{sugar} | \text{chocolate})$

s) $P(\text{waffle} | \text{vanilla})$

t) $P(\text{waffle} | \text{chocolate})$

Examples: Use the Venn diagram to find the following probabilities.



a) $P(\text{job}|\text{male})$

b) $P(\text{female}|\text{job})$

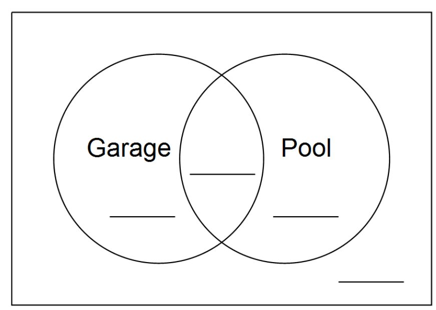
c) $P(\text{male}|\text{no job})$

d) $P(\text{no job}|\text{female})$

e) A student from the sample works at McTaco. What is the probability that the student is male?

f) Is a student from the sample more likely to have a job if he is a male? Justify your answer using conditional probability.

Examples: Real-estate ads suggest that 64% of homes have a garage, 21% have a pool, and 17% have both a garage and a pool. Fill in the Venn diagram, then answer the following questions.



a) Find $P(\text{garage} \cup \text{pool})$

b) Find $P(\text{garage}|\text{pool})$

c) Find $P(\text{pool}|\text{garage})$

d) Find $P(\text{pool}|\text{no garage})$

e) Find $P(\text{no pool}|\text{garage})$

f) Find $P(\text{no garage}|\text{no pool})$