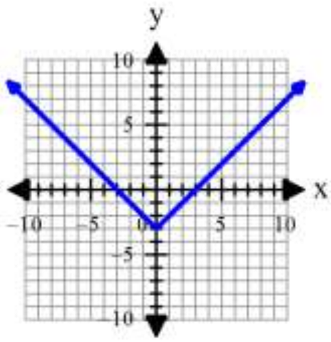
Name: ______________________________ Period: ___________

SM3 Unit 9 Logarithm Notes

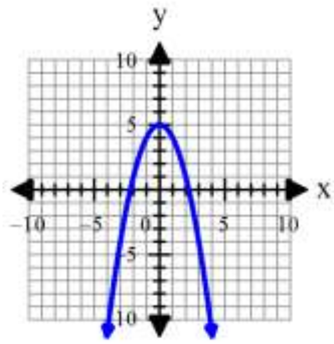
9.1 Notes – Inverse Functions

A. One-to-one functions

- Determine if the graphs below represents a function, a one-to-one function, or not a function.



#1



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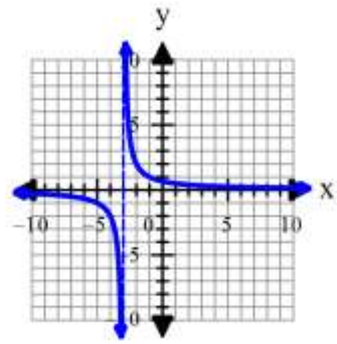


#3

C. Graphing inverses

Examples: Use the table of the relation to create the table of the relation's inverse.

x	f(x)
0	1
5	2
10	-3
15	-10
20	-16

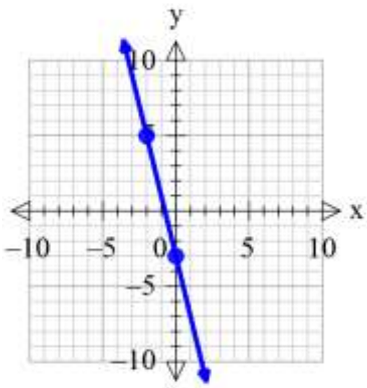
x	$f^{-1}(x)$

x	f(x)
-8	0.6
-6	0.8
-4	1
-2	1.2
0	1.4

x	$f^{-1}(x)$

Examples: Label each given point. Then graph the inverse of the point and label it. Draw the line of reflection and label it $y = x$. Draw the inverse of the graph. Be sure to label the new graph $f^{-1}(x)$.

1.



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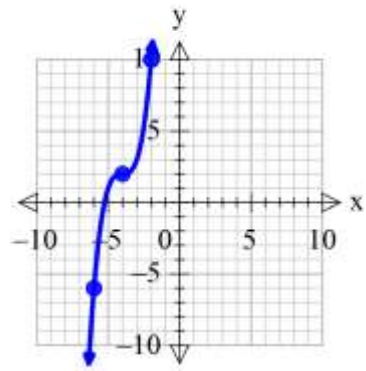


D. Finding the inverse of an equation.

Examples: Find $f^{-1}(x)$

1. $f(x) = 1 - 3x$

2. $f(x) = x^3 - 1$

3. $f(x) = \dfrac{-3x-4}{x-2}$

4. $f(x) = \dfrac{\sqrt{x-3}}{4}$

5. $f(x) = 2\sqrt[3]{x-2} - 4$

E. Are two functions inverses?

Examples: Check to see if $f(x)$ and $g(x)$ are inverses of each other. Must show work!

1. $f(x) = 3x + 4$ and $g(x) = \dfrac{x-4}{3}$

2. $f(x) = x^3 - 8$ and $g(x) = \sqrt[3]{x+8}$

9.2N – Exponents Review and Solving by Changing Base

A. Basic Properties of Exponents

1.	$b^o = 1$	Zero Property	1) $11^0 =$ _____
2.	$b^{-n} = \frac{1}{b^n}$ or $\frac{1}{b^{-n}} = b^n$	Negative Exponent Property	1) $5^{-3} =$ _____ 2) $\frac{1}{2^{-3}} =$ _____ $=$ _____ 3) $\left(\frac{1}{6}\right)^{-2}$ 4) $9 = 3^{\square} = \left(\frac{\square}{\square}\right)^{\square}$
3.	$(b^m)(b^n) = b^{m+n}$	Product Rule	1) $x^6 x^8 =$ _____
4.	$\frac{b^m}{b^n} = b^{m-n}$	Quotient Rule	1) $\frac{x^4}{x^2} =$ _____ 2) $\frac{x^6}{x^7} =$ _____
5.	$(b^m)^n = b^{m \cdot n}$	Power to a Power Rule	1) $(4x)^2 =$ _____ 2) $4x^2 =$ _____
6.	$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	Positive Rational Exponents	1) $16^{\frac{3}{2}} =$ _____ 2) $\frac{1}{8^{-\frac{4}{3}}} =$ _____

B. Write numbers as exponents.

Example: $9 = 3^2$ Hint: They all have more than one answer.	1. 4 =	2. 16 =	3. 32 =	4. 27 =	5. 243 =
	6. $\frac{1}{25} =$	7. $\frac{1}{2} =$	8. $\frac{1}{6^x} =$	9. 81 =	10. $\frac{1}{7} =$

C. Same base

- In the expression, 5^2: **5** is the __________ and **2** is the ______________.
- If the *bases* of both sides of an exponential equation are the **same**:

$$B^m = B^n$$

then

the *exponents* are *equal*: $m = n$

D. Steps to Solve by changing the base

$5^{3x} = \frac{1}{125}$	Given
$5^{3x} = \frac{1}{5^3}$	Express the denominator of the right side with a base of 5. We have 125 = 5^3. Apply the Negative Exponent Property.
$5^{3x} = 5^{-3}$	At this point, the bases are the same. Set the exponents equal to each other.
$3x = -3$	Solve for x.
$\frac{\cancel{3}x}{\cancel{3}} = \frac{-3}{3}$	To solve x, divide both sides by 3. That's it.
$x = -1$	

E. Examples

1. $4^5 = 4^x$	2. $7^{-3x-5} = 7^{2x}$	3. $3^{-3n} = 243$
4. $5^{-3x-3} = \frac{1}{625}$	5. $16^{m+1} = 64$	6. $81^{m+2} = \frac{1}{9}$
7. $\left(\frac{1}{9}\right)^{-3r-2} = 27^r$	8. $\frac{4^{-x}}{4^{5x-2}} = 32$	9. $\frac{16}{2^{2n+1}} = 8$

9.3 N – Exponential Functions

A. Warm-up: Practice ***Laws of Exponents***.

$a^s \cdot a^t =$ $\qquad$ $(a^s)^t =$ $\qquad$ $(ab)^s =$ $\qquad$ $1^s =$ $\qquad$ $a^{-s} =$ $\qquad$ $a^0 =$

B. Properties of Exponential Functions

An ***exponential function*** is a function of the form________________ where *a* is a positive real number $(a > 0)$ and $a \neq 1$. The domain of f is the set of all real numbers.

Examples: Determine if the functions below are exponential and explain why or why not.

x	$f(x)$
-1	12
0	4
1	4/3
2	4/9
3	4/27

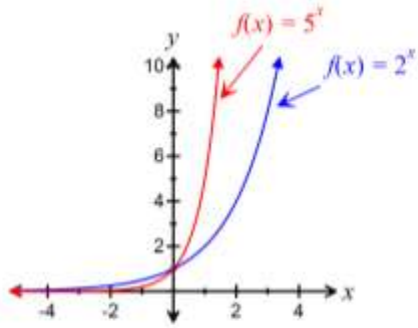
x	$f(x)$
-1	2
0	5
1	8
2	11
3	14

x	$f(x)$
-1	2/3
0	1
1	3/2
2	9/4
3	27/8

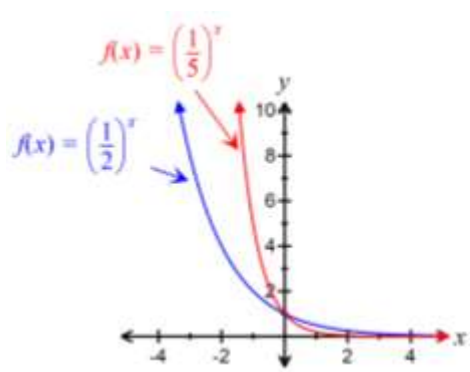
Properties of the Exponential Function $f(x) = a^x,\ a > 0,\ a \neq 1$

- Domain: ____________ Range: ______________
- There are no _______________; the *y*-intercept is _________.
- The *x*-axis $(y = 0)$ is a _______________________.
    - For $a > 1$, the graph approaches the *x*-axis as __________________
    - For $0 < a < 1$, the graph approaches the *x*-axis as _________________
- $f(x) = a^x$ is one-to-one.
    - For $a > 1$, $f(x) = a^x$ is an _________________ function.
    - For $0 < a < 1$, $f(x) = a^x$ is a ________________ function.
- The graph of f contains the points ______________, ______________, and ______________.

$a > 1$

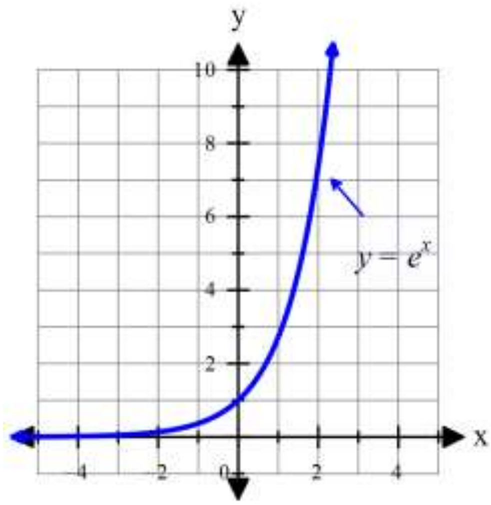


$0 < a < 1$



C. The number e

- The **number *e*** (approximately 2.71828…) is defined as the number that the expression $\left(1+\dfrac{1}{n}\right)^n$ approaches as $n \to \infty$. In calculus, this is expressed using limit notation as $e = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.



- Find e^2

D. Review Transformations. (No Calculators!)

- The general equation for an exponential function is: $y = b \cdot a^{c(x-h)} + k$
  List the transformation that corresponds with each variable.

  b =________________ c = ______________ h = _______________ k = _______________

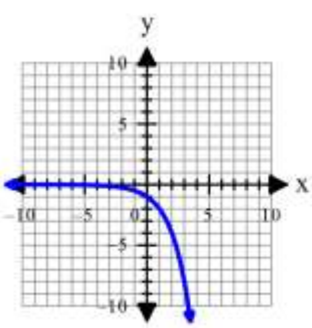
  Negative function ____________________ Negative exponent ___________________

- Without a Calculator, match each equation to the appropriate graph.

$a)\ \ y = 2^x$ $b)\ \ y = -2^x$ $c)\ \ y = 2^{-x}$ $d)\ \ y = 2^x - 1$

$e)\ \ y = -2^{-x}$ $f)\ \ y = 2^{x-1}$ $g)\ \ y = 1 - 2^x$ $h)\ y = 2^{1-x}$

1.



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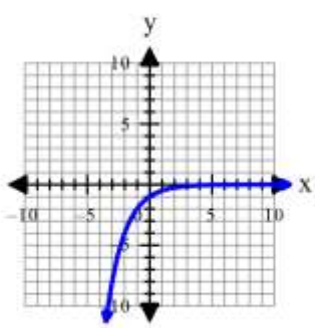

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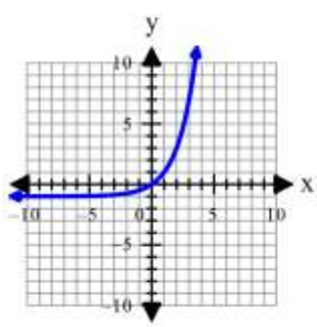

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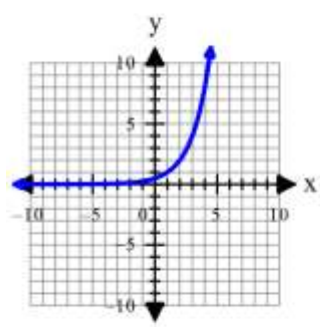

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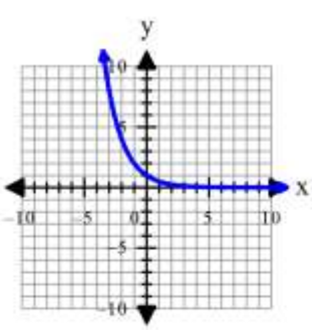

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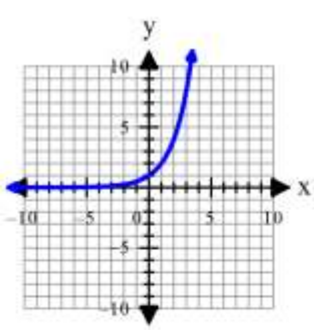

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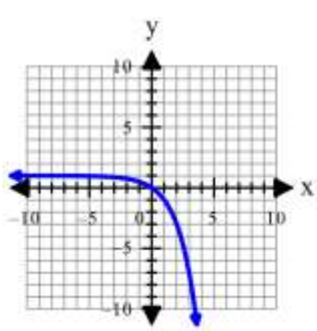

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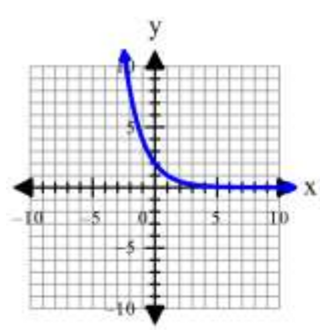
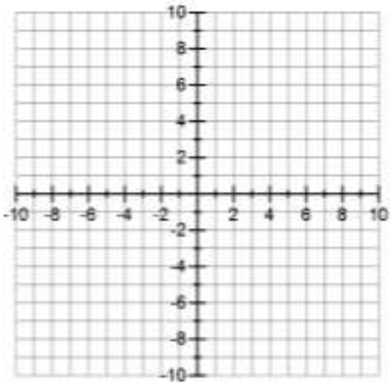


E. Graphing using transformations and 3 key points.

Examples: Use 3 key points and transformations to graph. (No Calculators!) Find domain, range, and horizontal asymptote.

a) Graph $f(x)=3^x$.



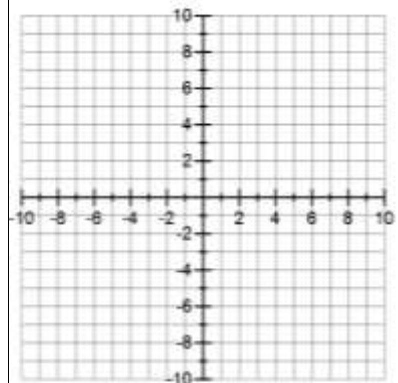
Domain:

Range:

Horizontal asymptote:

Key points and transformations:

b) Graph $f(x)=2\cdot\left(\frac{1}{3}\right)^x$.



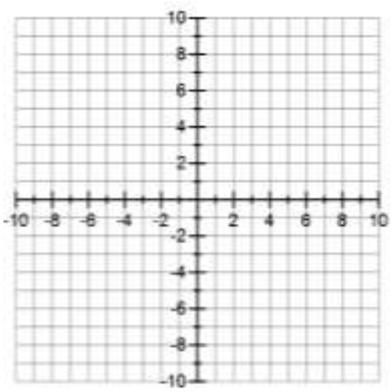
Domain:

Range:

Horizontal asymptote:

Key points and transformations:

c) Graph $f(x)=5^{x+3}$.



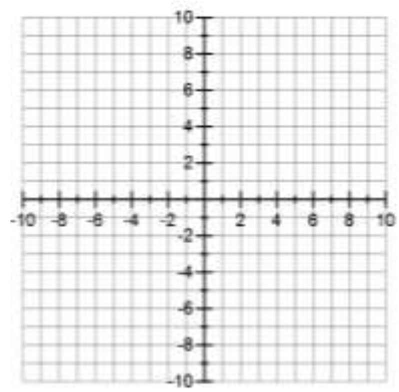
Domain:

Range:

Horizontal asymptote:

Key points and transformations:

d) Graph $f(x)=\left(\frac{1}{2}\right)^x+3$.



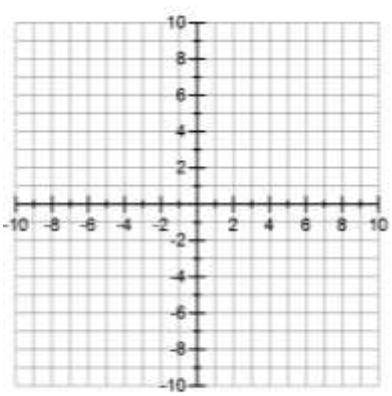
Domain:

Range:

Horizontal asymptote:

Key points and transformations:

e) Graph $f(x)=2^{-x}$.



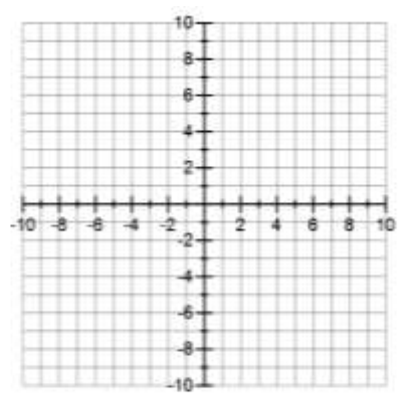
Domain:

Range:

Horizontal asymptote:

Key points and transformations:

f) Graph $f(x)=-3^x$



Domain:

Range:

Horizontal asymptote:

Key points and transformations:

9.4N – Logarithmic Functions

A. Inverses of exponential functions.

- The logarithmic function $y = \log_a x$ is the inverse of the exponential function ________.
- Domain $y = a^x$: _______. Range $y = a^x$:________.
- Domain $y = \log_a x$: _________. Range $y = \log_a x$:__________.

  Domain of the logarithmic function =

  _______ of the exponential function $= (0, \infty)$

  Range of the logarithmic function =

  _________ of the exponential function $= (-\infty, \infty)$

★ **Caution!** You can't take the log of zero or of a negative because it is impossible to get zero or a negative by raising a positive base to an exponent. **The argument of a logarithmic function must be greater than zero.**

Properties of the Logarithmic Function $f(x) = \log_a x$

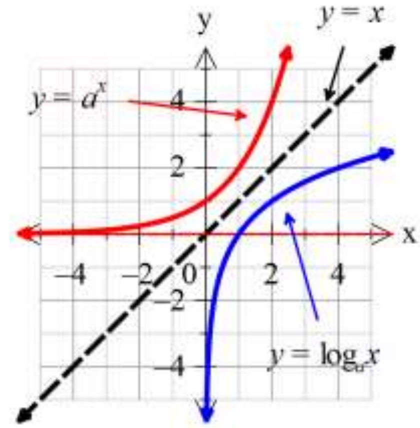
- The x-intercept is _____. There is _____ y-intercept.
- The vertical asymptote of the graph is _________________.
- The logarithmic function is ____________ if $0 < a < 1$ and ______________ if $a > 1$. The function is one-to-one.
- Since $y = \log_a x$ is the inverse of $y = a^x$ and the graph $y = a^x$ contains the points $\left(-1, \frac{1}{a}\right)$, $(0,1)$, and $(1, a)$ then the graph of $y = \log_a x$ contains the points $(\_\_\_\_, \_\_\_\_), (\_\_\_\_, \_\_\_\_)$, and $(\_\_\_\_, \_\_\_\_)$.

Common Logarithmic Function: If the base of a logarithmic function is the number 10, then we have the common logarithmic function. If the base a of the logarithmic function is not indicated, it is understood to be 10. That is, $y = \log x$ if and only if $x = 10^y$.

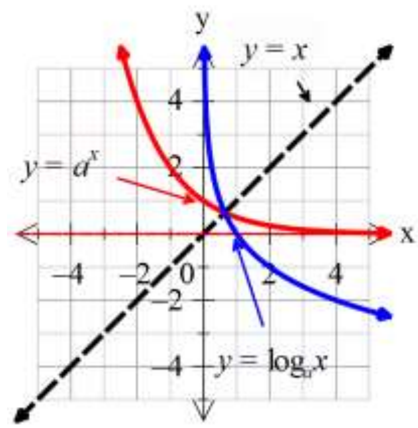
Natural Logarithms: If the base of a logarithmic function is the number *e*, then we have the natural logarithm function (abbreviated ln). That is, $y = \ln x$ if and only if $x = e^y$.

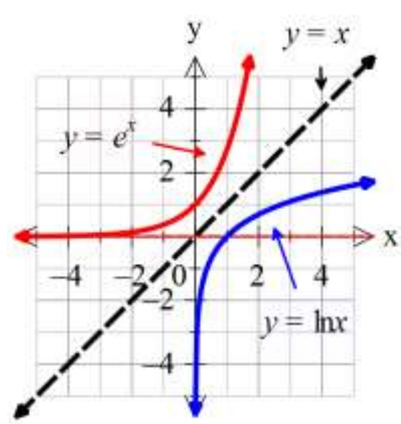
- $y = \ln x$ is the ____________of $y = e \wedge x$

$a > 1$



$0 < a < 1$





B. Finding the domain of logarithmic functions.

1. $f(x) = \log_2(x+3)$
2. $h(x) = -\log_{\frac{1}{2}} x$
3. $g(x) = \ln(-x-5)$

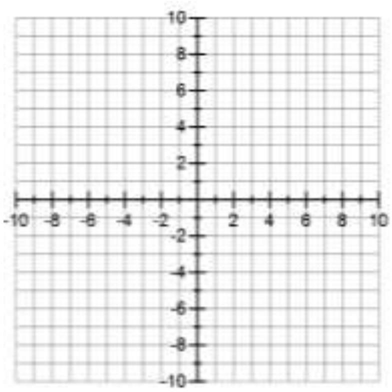
C. Graphing logarithmic functions.

Steps for Graphing Logarithmic Functions:

1. Find the domain
2. Find the asymptotes
3. Graph the asymptotes
4. Find the 3 key points $(1,0)$, $(a,1)$, **and** $\left(\frac{1}{a}, -1\right)$ and apply the appropriate transformations.
5. Plot your points and connect them to form a smooth curve.
6. Find the range

Examples: Graph the following functions.

a) $y = 2^x$ and $y = \log_2 x$



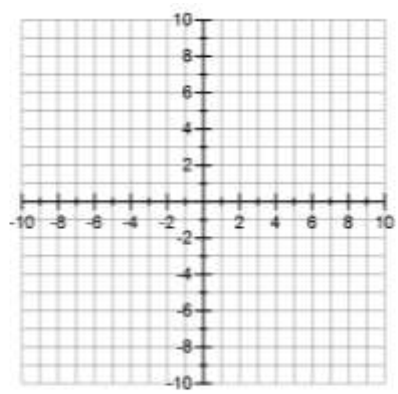
Domain:

Asymptotes:

Key points and transformations:

Range:

b) $y = \log(-x) - 2.$



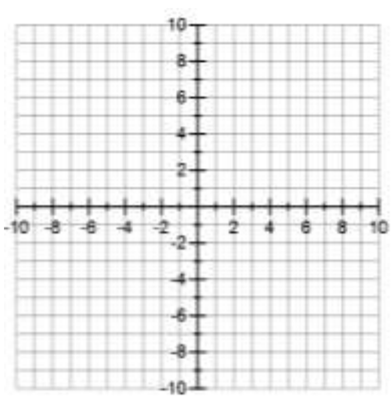
Domain:

Asymptotes:

Key points and transformations:

Range:

c) $f(x) = -\ln(x+3)$



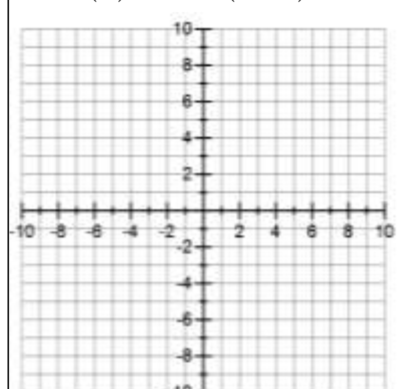
Domain:

Asymptotes:

Key points and transformations:

Range:

d) $f(x) = 2\log(x-3)$



Domain:

Asymptotes:

Key points and transformations:

Range:

D. Finding the inverse of a logarithmic function.

- **$\log_2 x$ means "the exponent to which we raise 2 to get *x*."**
  Pronounced "the logarithm, base 2, of *x*" or "log, base 2, of *x*"

★LOGARITHMS ARE EXPONENTS!★

- ***Logarithm:*** $\log_b a$ means the ***exponent*** to which we raise b to get a.
  b is called the ***base*** of the logarithm (the number being raised to the exponent).
  a is called the ***argument*** of the logarithm (the number you get when you raise the base to the exponent).

The ***logarithmic function of base b***, where $b>0 \text{ and } b\neq 1$ is denoted by $y=\log_b x$ and is defined by

$$y=\log_b x \text{ if and only if } x=b^y.$$

Example: Change each exponential expression to an equivalent expression involving a logarithm.

a) $5^x=625$ b) $x^3=64$ c) $3^2=x$

Example: Change each logarithmic expression to an equivalent expression involving an exponent.

a) $\log_3 x=5$ b) $\log_e 5=x$ c) $\log_m 2=n$

E. Evaluating Logarithms

- Instead of "$\log_2 8=x$," think, what power of 2 equals 8? Or 2 to what power equals 8?
  - $2^x=8$
  - The answer would be 3 because $2^3=8$.

Example: Find the exact value of each logarithm without using a calculator.

a) $\log_3 9=x$ b) $\log_2 32=x$ c) $\log_6 1=x$ d) $\log_5 \frac{1}{125}=x$ e) $\log_7 \sqrt{7}=x$

9.5 N – Solving Logarithmic Equations

A. Review

Change each logarithmic statement into an equivalent exponential statement.

1. $\log_8 64 = 2$
2. $\log_2 \frac{1}{16} = -4$
3. $\log 8 = x$
4. $\ln x = 5$

Change each exponential statement into an equivalent logarithmic statement.

1. $4^x = 27$
2. $3^{-4} = \frac{1}{81}$
3. $9^x = 3.2$

Solve the following equation using the laws of exponents.

1. $16^{m+2} = 64$
2. $9^{-3n} = 243$

B. Solving Logarithmic and Exponential Equations

- Use the properties of logarithms and exponents to manipulate the equations.
    a. Remember the *exponential* property: $a^u = a^v \Leftrightarrow u = v$.
- Try rewriting as an *exponential* function: $y = \log_a x \Leftrightarrow x = a^y$ **or**
  as a *logarithmic* equation: $x = a^y \Leftrightarrow y = \log_a x$

Examples:

a) $\log_{18} 324 = x$

b) $6^{x-4} = 11$

c) $\ln e^{2x} = 6$

d) $3 \cdot (10)^{3-x} = 7$

e) $\log_3(3x-1) = 2$

f) $2^{-x} = 1.5$

g) $\log_6 216 = 3x + 2$

h) $e^{4x+3} = 9$

9.6 – Properties of Logarithms

- ***Remember:*** *Definition of Logarithm*: $y = \log_a x \;\Leftrightarrow\; a^y = x$

A. Properties of Logarithms

For any positive numbers M, N, and a, where $a \neq 1$ and r is any real number:

If $a^0 = 1$ then
- $\log_a 1 =$ _____

If $a^1 = a$ then
- $\log_a a =$ _____

If $a^M = \log_a M$ then
- $a^{\log_a M} =$ _____

If $a^r = a^r$ then
- $\log_a a^r =$ _____

- $\log_a (MN) = \log_a M + \log_a N$
- $\log_a M^r = r \log_a M$
- $\log_a \left(\dfrac{M}{N}\right) = \log_a M - \log_a N$
- $\log_a M = \log_a N \;\Leftrightarrow\; M = N$

Change of Base Formula:
- $\log_a M = \dfrac{\log_b M}{\log_b a}$
- $\log_a M = \dfrac{\log M}{\log a}$
- $\log_a M = \dfrac{\ln M}{\ln a}$

Examples: Find the exact value of each expression. (Do not use a calculator).

a) $\log_4 1$

b) $5^{\log_5 3}$

c) $\log_7 7^{-1}$

d) $\ln e$

e) $\log_2 64$

f) $\log_7 \dfrac{1}{\sqrt[3]{49}}$

Examples: Write each expression as a *sum/difference* of logarithms. Express powers as factors.

a) $\log 5x$

b) $\ln \dfrac{3}{x}$

c) $\log_7 \left(x^5\right)$

d) $\ln\left(x^2 e^x\right)$

e) $\log \dfrac{\sqrt[4]{x}}{\sqrt[4]{y}}$

f) $\ln \dfrac{y^4}{x^5}$

Examples: Write each expression as a single logarithm.

a) $\ln 8 + \ln x$

b) $\log u - \log v$

c) $\frac{1}{4}\log x$

d) $\log_7 u + 3\log_7 v$

e) $4\ln(uv) - 3\ln(vw)$

f) $\log(x-4) + \log(6x+5)$

Examples: Use the *change of base* formula to evaluate each logarithm.

a) $\log_6 9$

b) $\log_{\sqrt{2}} 7$

c) $\log_\pi \sqrt{3}$

Examples: Write the expression using only *natural logarithms.*

a) $\log_7 30$

b) $\log_4 10$

Examples: Write the expression using only *common logarithms*.

a) $\log_6 y$

b) $\log_2(d+e)$

Examples: Use properties of logarithms to find the exact value of each expression. (Do not use a calculator).

a) $\log_7 21 - \log_7 3$

b) $5^{\log_5 6 + \log_5 7}$

c) $\log_4 11 \cdot \log_{11} 256$

9.7 N - Solving Logarithmic Equations

A. Review

1) $\log_3 x = 4$

2) $27\left(\frac{1}{3}\right)^{x/5} = 3$

3) $16 \cdot 4^{x/3} = 1024$

B. Use the Properties of Logarithms and Exponents to solve equations.

- To avoid extraneous solutions, determine the domain of the variable first.
- Use the *properties of logarithms and exponents* to manipulate the equations.
- Try rewriting as an exponential or logarithmic function: $y = \log_a x \Leftrightarrow x = a^y$
- Remember the properties: $\log_a M = \log_a N \Leftrightarrow M = N$ and $a^u = a^v \Leftrightarrow u = v$ (Make the bases the same).
- Check your solution by substituting into the original equation.

a) $4^x = 37$

b) $2.05^x = 4.36$

c) $30e^{0.014x} = 600$

d) $8 - 5e^{-x} = -12$

e) $2^{4-x} - 7 = 14$

f) $\log_4 x = \log_4(3x - 8)$

g) $\ln x^2 = 8$

h) $-4\log(x+5) - 3 = -4$

i) $-2\log_4 x = \log_4 9$

j) $3\log_2(x-1) + \log_2 4 = 5$

k) $\ln(5x) - \ln(10) = 5$

l) $\log_6(x+4) + \log_6(x+3) = 1$

m) $\log_2 x + \log_2(x-2) = \log_2(x+4)$

9.8N - Financial Models & Exp. Growth & Decay Models

A. Review

Express each percent as a decimal.

1) 3% 2) 13.5% 3) 102%

B. *Simple Interest*: If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is $\boldsymbol{I = Prt}$.

Example1: What is the interest due if $1000 is borrowed for 9 months at a simple interest rate of 5% per year?

Example2: If you borrow $7000 and, after 6 months pay off the loan in the amount of $7,500, what yearly rate of interest was charged?

C. *Compound Interest:* When the interest due at the end of a payment period is added to the principal so that the interest computed at the end of the next payment period is based on the new principal amount (old principal plus interest), the interest is said to have been ***compounded***. ***Compound interest*** is interest paid on principal and previously earned interest.

The amount A after t years due to a principal P invested at an annual interest rate t compounded n times per year is $\boldsymbol{A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}}$. *Present value* or to find the principal: $\boldsymbol{P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt}}$.

Example 1: Investing $1000 at an annual rate of 9% compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 1 year:

Annual Compounding (n = 1):

Semiannual Compounding (n = 2):

Quarterly Compounding (n = 4):

Monthly Compounding (n = 12):

Daily Compounding (n = 365):

Example 2: How much money must be invested now in order to end up with $20,000 in 10 years at 5% compounded quarterly?

D. ***Continuous Compounding***

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is $\boldsymbol{A = Pe^{rt}}$. *Present value* or to find the Principal: $\boldsymbol{P = Ae^{-rt}}$.

Example 1: Find the amount A that results from investing a principle P of $1000 at an annual rate r of 9% compounded continuously for a time t of 1 year.

Example 2: How much money must be invested now in order to end up with $20,000 in 10 years at 3.8% compounded continuously?

E. Exponential Growth and Decay Models

***Law of Uninhibited Growth or Decay*:**
Many natural phenomena have been found to follow the law that an amount A varies with time t according to the function $A(t) = A_0 e^{kt}$, where A_0 is the original amount at time $t = 0$ and k is a constant of growth or decay (growth if $k > 0$, decay if $k < 0$.)

Example 1: The number N of bacteria present in a culture at time t hours obeys the law of uninhibited growth where $N(t) = 1000e^{0.01t}$.

a) Determine the number of bacteria at $t = 0$ hours.

b) What is the growth rate of the bacteria?

c) What will the population be after 4 hours?

d) When will the number of bacteria reach 1700?

e) When will the number of bacteria double?

Example 2: Iodine 131 is a radioactive material that decays according to the function $A(t) = A_0 e^{-0.087t}$, where A_0 is the initial amount present and A is the amount present at time t (in days). Assume that a scientist has a sample of 100 grams of iodine 131.

a) What is the decay rate of iodine 131?

b) How much iodine 131 is left after 9 days?

c) When will 70 grams of iodine 131 be left?

d) What is the half-life of iodine 131? (when $A = \frac{1}{2} A_0$.)