Name: ______________________________ Period: ___________

SM3 Unit 9 Logarithm Notes

9.1 Notes – Inverse Functions

Function: For every x, there is one y.

1:1

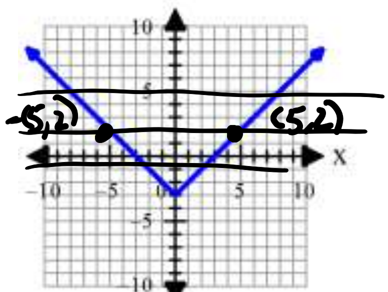
A. One-to-one functions:

For every y there is one x.

* A function has an inverse only if it is 1:1

Determine if the graphs below represents a function, a one-to-one function, or not a function.

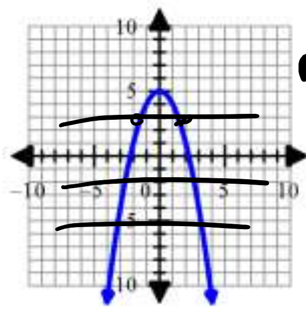
* If it passes the vertical line test, it is a function.



not 1:1

Function

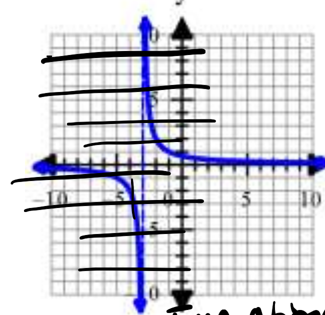
#1



not 1:1

Function

#2



1:1 passes horizontal line test

Function

#3

* If it passes the horizontal line test, then it is a one to one function

* If it passes horizontal line test, the function has an inverse.

$f^{-1}(x)$ inverse function

* Inverse function as all same points as original function except the x & y are reversed.

C. Graphing inverses

Examples: Use the table of the relation to create the table of the relation's inverse.

Function

x	f(x)
0	1
5	2
10	-3
15	-10
20	-16

Inverse

x	$f^{-1}(x)$
1	0
2	5
-3	10
-10	15
-16	20

Function

x	f(x)
-8	0.6
-6	0.8
-4	1
-2	1.2
0	1.4

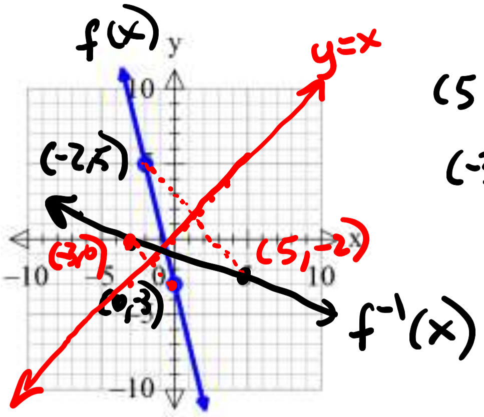
x	$f^{-1}(x)$
.6	-8
.8	-6
1	-4
1.2	-2
1.4	0

✓Reverse x & y values

Examples: Label each given point. Then graph the inverse of the point and label it. Draw the line of reflection and label it $y = x$. Draw the inverse of the graph. Be sure to label the new graph $f^{-1}(x)$.

1.

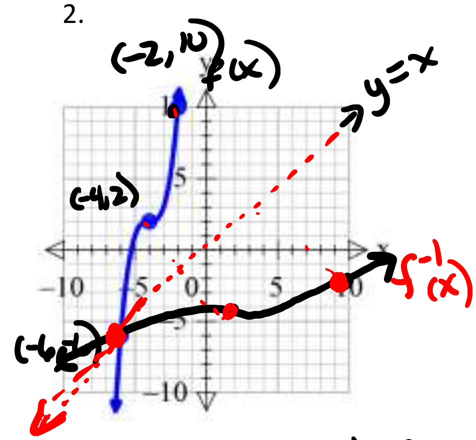
$y = \frac{1}{1}x + 0$



(5,-2)

(-3,0)

2.



(10,-2)

(2,-4)

(-6,-6)

Steps to find inverse of function
① replace f(x) with y
② swap x & y
③ Solve for y
④ write in inverse notation

D. Finding the inverse of an equation.

Examples: Find $f^{-1}(x)$

1. $f(x) = 1 - 3x$

$y = 1 - 3x$

$x = 1 - 3y$

$-1 \quad -1$

$\frac{x-1}{-3} = \frac{-3y}{-3}$

$y = \frac{x-1}{-3}$

$\boxed{f^{-1}(x) = \frac{x-1}{-3}}$ or $\boxed{f^{-1}(x) = -\frac{1}{3}x + \frac{1}{3}}$

2. $f(x) = x^3 - 1$

$y = x^3 - 1$

$x = y^3 - 1$

$\sqrt[3]{x+1} = \sqrt[3]{y^3}$

$\sqrt[3]{x+1} = y$

$y = \sqrt[3]{x+1}$

$\boxed{f^{-1}(x) = \sqrt[3]{x+1}}$

3. $f(x) = \frac{-3x-4}{x-2}$

$y = \frac{-3x-4}{x-2}$

$(y-2)x = \frac{-3y-4}{(y-2)} \cdot (y-2)$

$xy - \cancel{2x} = -\cancel{3y} - 4$

$+3y \quad +2x \qquad +3y \quad +2x$

$xy + 3y = 2x - 4$

$\frac{y(x+3)}{(x+3)} = \frac{2x-4}{x+3}$

$y = \frac{2x-4}{x+3}$

$\boxed{f^{-1}(x) = \frac{2x-4}{x+3}}$

4. $f(x) = \frac{\sqrt{x-3}}{4}$

$y = \sqrt{x-3}$

$x^2 = (\sqrt{y-3})^2$

$x^2 = y - 3$

$+3 \quad +3$

$y = x^2 + 3$

$\boxed{f^{-1}(x) = x^2 + 3}$

5. $f(x) = 2\sqrt[3]{x-2} - 4$

$y = 2\sqrt[3]{x-2} - 4$

$x = 2\sqrt[3]{y-2} - 4$

$+4 \qquad +4$

$\frac{x+4}{2} = \frac{2\sqrt[3]{y-2}}{2}$

$\left(\frac{x+4}{2}\right)^3 = \left(\sqrt[3]{y-2}\right)^3$

$\frac{(x+4)^3}{8} = y - 2$

$+2 \qquad +2$

$y = \frac{(x+4)^3}{8} + 2$

$\boxed{f^{-1}(x) = \frac{(x+4)^3}{8} + 2}$

E. Are two functions inverses?

Examples: Check to see if $f(x)$ and $g(x)$ are inverses of each other. Must show work!

$f(g(x)) = g(f(x)) = x$

1. $f(x) = 3x + 4$ and $g(x) = \frac{x-4}{3}$

$f\left(\frac{x-4}{3}\right) = \cancel{3}\left(\frac{x-4}{\cancel{3}}\right) + 4$

$= x - \cancel{4} + \cancel{4}$

$= \boxed{x}$

Inverses

$g(3x+4) = \frac{3x + \cancel{4} - \cancel{4}}{3}$

$= \frac{3x}{3} = \boxed{x}$

2. $f(x) = x^3 - 8$ and $g(x) = \sqrt[3]{x+8}$

$f(\sqrt[3]{x+8}) = (\sqrt[3]{x+8})^3 - 8$

$x + \cancel{8} - \cancel{8}$

$\boxed{x}$

$g(x^3 - 8) = \sqrt[3]{x^3 - \cancel{8} + \cancel{8}}$

$= \cancel{\sqrt[3]{x^3}}$ $\boxed{x}$

9.2N – Exponents Review and Solving by Changing Base

A. Basic Properties of Exponents

1.	$b^0 = 1$	Zero Property	1) $11^0 = \underline{1}$
2.	$b^{-n} = \frac{1}{b^n}$ or $\frac{1}{b^{-n}} = b^n$	Negative Exponent Property	1) $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$ 2) $\frac{1}{2^{-3}} = \underline{2^3} = \underline{8}$ 3) $\left(\frac{1}{6}\right)^{-2} = \frac{1^{-2}}{6^{-2}} = \frac{6^2}{1^2} = 36$ 4) $9 = 3^{\boxed{2}} = \cancel{\left(\frac{\square}{\square}\right)^{\square}}$
3.	$(b^m)(b^n) = b^{m+n}$	Product Rule	1) $x^6 \cdot x^8 = \underline{x}^{6+8} = x^{14}$
4.	$\frac{b^m}{b^n} = b^{m-n}$	Quotient Rule	1) $\frac{x^{4-2}}{\cancel{x}} = \underline{x^2}$ 2) $\frac{\cancel{x^6}}{x^{7-6}} = \frac{1}{x^1} = \frac{1}{x}$
5.	$(b^m)^n = b^{m \cdot n}$	Power to a Power Rule	1) $(4x)^2 = 4^2x^2 = 16x^2$ 2) $4x^2 = \underline{4x^2}$
6.	$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	Positive Rational Exponents	1) $16^{\frac{3}{2}} = \left(\sqrt[2]{16}\right)^3$ $4^3 = 64$ 2) $\frac{1}{8^{-\frac{4}{3}}} = \underline{8}^{4/3} = \left(\sqrt[3]{8}\right)^4$ $2^4 = 16$

B. Write numbers as exponents.

Example: $9 = 3^2$ Hint: They all have more than one answer.	1. $4 = 2^2$	2. $16 = 4^2$ $16 = 2^4$	3. $32 = 2^5$ $32 = \frac{1}{2^{-5}} = 2^5$	4. $27 = 3^3$ $27 = \frac{1}{3^{-3}}$	5. $243 = 3^5$ or $243 = \frac{1}{3^{-5}}$
	6. $\frac{1}{25} = \frac{1}{5^2} = 5^{-2}$	7. $\frac{1}{2} = \frac{1}{2^1} = 2^{-1}$	8. $\frac{1}{6^x} = 6^{-x}$	9. $81 = \underline{9}^2 = \frac{1}{9^{-2}}$ $81 = \underline{3}^4$	10. $\frac{1}{7^1} = 7^{-1}$

C. Same base

- In the expression, 5^2: **5** is the $\underline{\text{base}}$ and **2** is the $\underline{\text{exponent}}$
- If the *bases* of both sides of an exponential equation are the **same**:

$$\underline{B}^m = \underline{B}^n$$

then

the *exponents* are *equal*: $m = n$

Example $\underline{2}^5 = \underline{2}^5$ exp. equal $5 = 5$

* If bases are same, the exponents must be the same

D. Steps to Solve by changing the base

5^{-3} $5^{3x} = \frac{1}{125}$	Given
$5^{3x} = \frac{1}{5^3}$	Express the denominator of the right side with a base of 5. We have 125 = 5^3.
$5^{3x} = 5^{-3}$ exp. are same since bases are same	Apply the Negative Exponent Property. At this point, the bases are the same. Set the exponents equal to each other.
$3x = -3$	Solve for x.
$\frac{\cancel{3}x}{\cancel{3}} = \frac{-3}{3}$	To solve x, divide both sides by 3. That's it.
$x = -1$	

* Trying to get same base so that you can set exponents equal and solve.

E. Examples

1. $4^5 = 4^x$ $\boxed{5=x}$ bases same so exp. same	2. $7^{-3x-5} = 7^{2x}$ $-3x-5=2x$ $+3x \quad +3x$ $\frac{-5}{5}=\frac{5x}{5}$ $\boxed{-1=x}$	3. $3^{-3n} = 243$ $3^{-3n}=3^5$ $\frac{-3n}{-3}=\frac{5}{-3}$ $\boxed{n=-5/3}$
4. $5^{-3x-3} = \frac{1}{625}$ $5-3x-3 = \frac{1}{5^4}$ $5^{-3x-3}=5^{-4}$ $-3x-3=-4$ $+3 \quad +3$ $\frac{-3x}{-3}=\frac{-1}{-3}$ $\boxed{x=\frac{1}{3}}$	5. $16^{m+1} = 64$ * change both bases $4^{2(m+1)}=4^3$ $2(m+1)=3$ $2m+2=3$ $-2 \quad -2$ $\frac{2m}{2}=\frac{1}{2}$ $\boxed{m=\frac{1}{2}}$	6. $81^{m+2} = \frac{1}{9}$ $9^{2(m+2)}=\frac{1}{9^1}$ $9^{2(m+2)}=9^{-1}$ $2(m+2)=-1$ $2m+4=-1$ $-4 \quad -4$ $\frac{2m}{2}=\frac{-5}{2}$ $\boxed{m=\frac{-5}{2}}$
7. $\left(\frac{1}{9}\right)^{-3r-2} = 27^r$ change base to 3 $\frac{1}{3^2}=3^{3r}$ $(3^{-2})^{-3r-2}=3^{3r}$ $-2(-3r-2)=3r$ $+6r+4=3r$ $-6r \quad -6r$ $\frac{4}{-3}=\frac{-3r}{-3}$ $\boxed{r=-\frac{4}{3}}$	8. $\frac{4^{-x}}{4^{5x-2}} = 32$ $4^{-x-5x+2}$ $4^{-x-5x+2}=32$ $4^{-6x+2}=32$ $2^{2(-6x+2)}=2^5$ $2(-6x+2)=5$ $-12x+4=5$ $-12x=1$ $\boxed{x=-1/12}$	9. $\frac{16}{2^{2n+1}} = 8$ $\frac{2^{4-2n-1}}{\cancel{2^{2n+1}}}=2^3$ $2^{4-2n-1}=2^3$ $2^{3-2n}=2^3$ $3-2n=3$ $-3 \quad -3$ $\frac{-2n}{-2}=\frac{0}{-2}$ $\boxed{n=0}$

9.3 N – Exponential Functions

A. Warm-up: Practice ***Laws of Exponents***.

$a^s \cdot a^t = a^{s+t}$ $\quad (a^s)^t = a^{s\cdot t}$ $\quad (a^1b^1)^s = a^s b^s$ $\quad 1^s = 1$ $\quad a^{-s} = \frac{1}{a^s}$ $\quad a^0 = 1$

Ex $x^3 \cdot x^5 = x^8$ $\quad$ Ex. $(x^4)^2 = x^8$ $\quad$ Ex. $(2x^2)^2$ $\quad 2^2x^4 = 4x^4$

B. Properties of Exponential Functions

An ***exponential function*** is a function of the form $y = a^x$ where a is a positive real number $(a > 0)$ and $a \neq 1$. The domain of f is the set of all real numbers. ↖ base

Examples: Determine if the functions below are exponential and explain why or why not.

x	$f(x)$
-1	$12 \div 3 = 4$
0	$4 \div 3$ $\quad 4/3$
1	$4/3 \cdot 1/3 = 4/9$
2	$4/9$
3	$4/27$

+

yes $\div$ by 3 or mult. by $1/3$

x	$f(x)$
-1	2
0	5
1	8
2	11
3	14

+3 +3 +3 +3

no linear

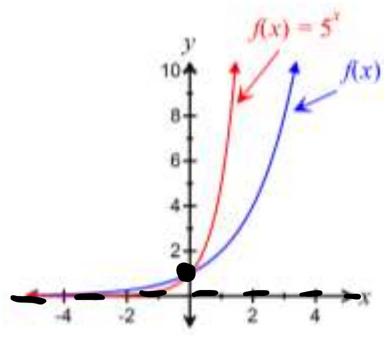
x	$f(x)$
-1	$2/3 \cdot 3/2$
0	$1 \cdot 3/2$
1	$3/2 \cdot 3/2$
2	$9/4$
3	$27/8$

yes mult. by $3/2$

Properties of the Exponential Function $f(x) = a^x$, $a > 0$, $a \neq 1$

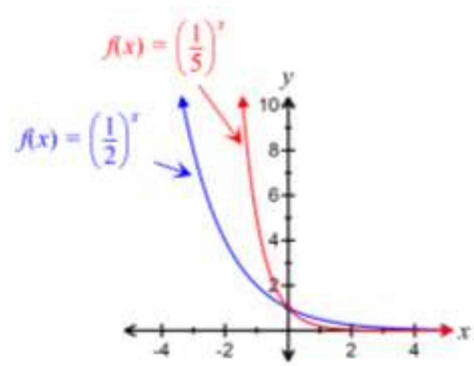
- Domain: $(-\infty, \infty)$ Range: $(0, \infty)$
- There are no x int.; the *y*-intercept is $(0,1)$
- The *x*-axis $(y = 0)$ is a asymptote.
    - For $a > 1$, the graph approaches the *x*-axis as x approaches $-\infty$
    - For $0 < a < 1$, the graph approaches the *x*-axis as x approaches ∞
- $f(x) = a^x$ is one-to-one.
    - For $a > 1$, $f(x) = a^x$ is an increasing and positive function.
    - For $0 < a < 1$, $f(x) = a^x$ is a decreasing and positive function.
- * The graph of f contains the points $(-1, 1/a)$, $(1, a)$, and $(0,1)$.

$a > 1$



y=0 ~~x~~ x axis

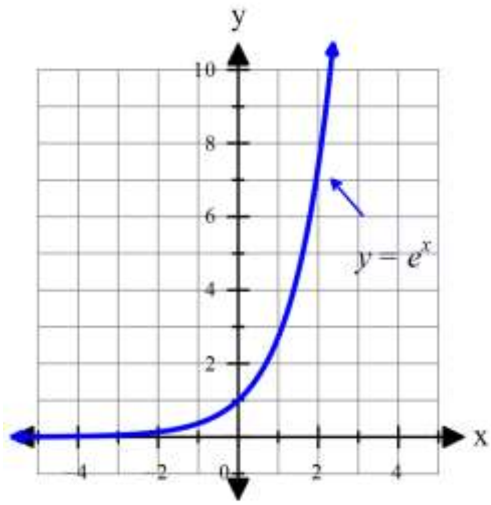
$0 < a < 1$



C. The number e

- The **number *e*** (approximately 2.71828...) is defined as the number that the expression $\left(1+\frac{1}{n}\right)^n$ approaches as $n \to \infty$. In calculus, this is expressed using limit notation as $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

- Find e^2



D. Review Transformations. (No Calculators!)

- The general equation for an exponential function is: $y = b \cdot a^{c(x-h)} + k$
  List the transformation that corresponds with each variable.

  b = vertical stretch or shrink c = horizontal stretch or shrink (opposite) h = horizontal shift left or right (opposite) k = Vertical shift up or down

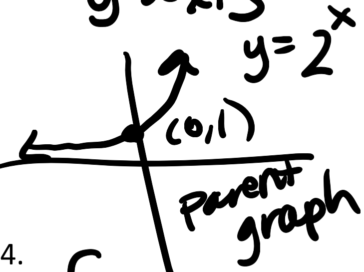
  Negative function Reflection over x axis b neg Negative exponent Reflection over y axis c neg

- Without a Calculator, match each equation to the appropriate graph.

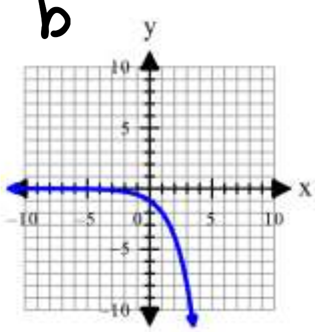
$\cancel{a)}\ y = 2^x$ $\cancel{b)}\ y = -2^x$ $\cancel{c)}\ y = 2^{-x}$ $\cancel{d)}\ y = 2^x - 1$

$y = -2^x + 1$

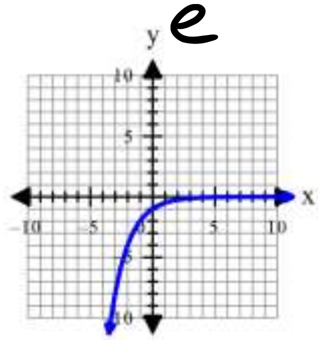
$\cancel{e)}\ y = -2^{-x}$ $\cancel{f)}\ y = 2^{x-1}$ $\cancel{g)}\ y = 1 - 2^x$ $h)\ y = 2^{1-x}$ $y = 2^{-x+1}$



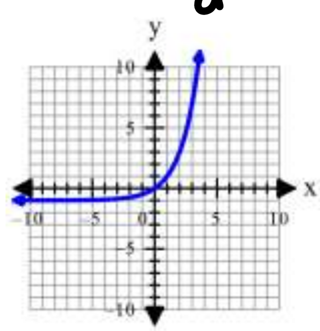
1. b



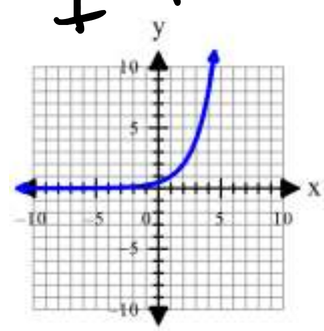
2. e



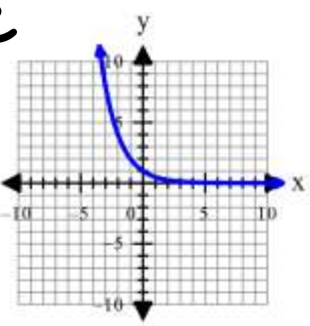
3. d



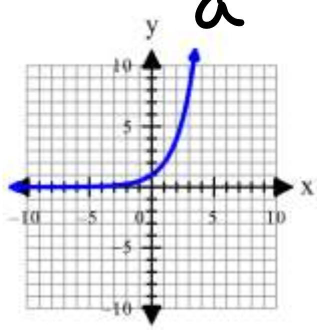
4. f



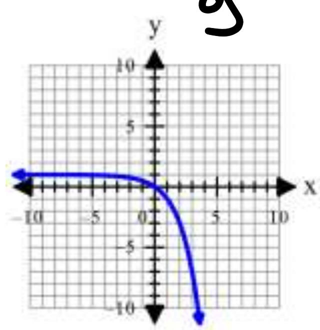
5. c



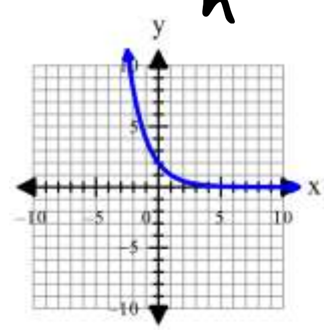
6. a



7. g



8. h



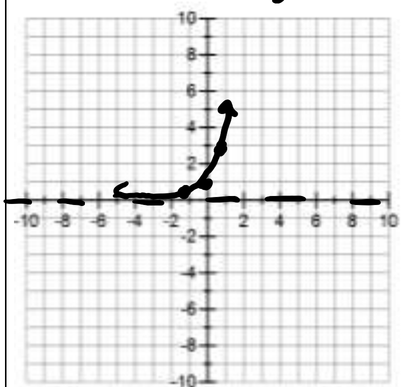
E. Graphing using transformations and 3 key points. key points *

x	y
-1	1/a ← reciprocal
0	1
1	a

* a is my base

Examples: Use 3 key points and transformations to graph. (No Calculators!) Find domain, range, and horizontal asymptote.

a) Graph $f(x)=3^x$. a=3



Domain: $(-\infty,\infty)$

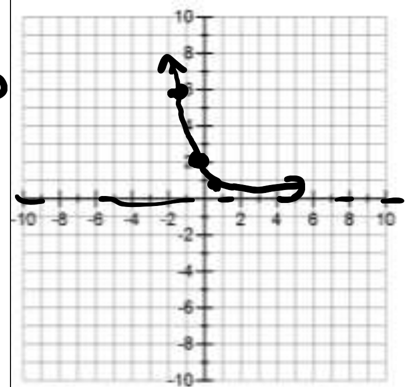
Range: $(0,\infty)$

Horizontal asymptote: x axis or y=0

Key points and transformations: key points

x	y
-1	1/3
0	1
1	3

b) Graph $f(x)=2\cdot\left(\frac{1}{3}\right)^x$. a=1/3



Domain: $(-\infty,\infty)$

Range: $(0,\infty)$

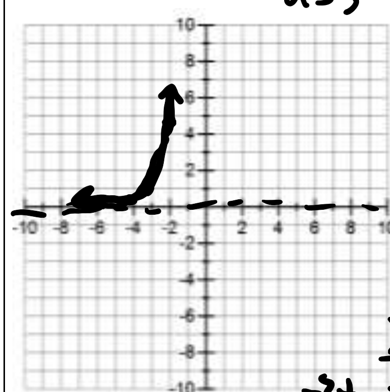
Horizontal asymptote: y=0 or x axis

Key points and transformations: vertical stretch of 2

x	y
-1	3·2
0	1·2
1	1/3·2

x	y
-1	6
0	2
1	2/3

c) Graph $f(x)=5^{x+3}$ ← opposite a=5



Domain: $(-\infty,\infty)$

Range: $(0,\infty)$

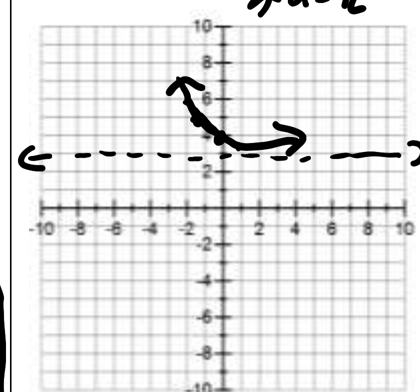
Horizontal asymptote: y=0 x axis

Key points and transformations: left 3

	x	y
-3+	-1	1/5
-3+	0	1
-3+	1	5

x	y
-4	1/5
-3	1
-2	5

d) Graph $f(x)=\left(\frac{1}{2}\right)^x+3$ a=1/2



Domain: $(-\infty,\infty)$

Range: $(3,\infty)$

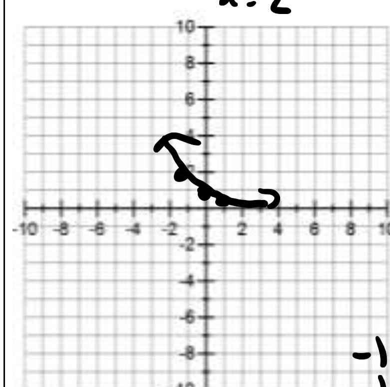
Horizontal asymptote: y=3

Key points and transformations: up 3

x	y
-1	2+3
0	1+3
1	1/2+3

x	y
-1	5
0	4
1	3.5

e) Graph $f(x)=2^{-x}$. a=2



Domain: $(-\infty,\infty)$

Range: $(0,\infty)$

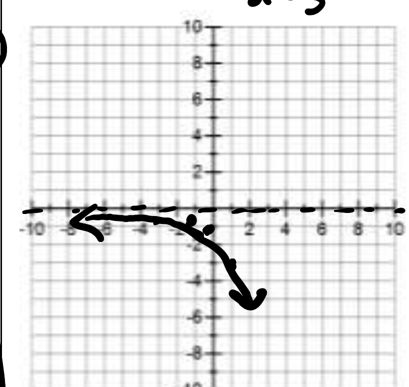
Horizontal asymptote: y=0

Key points and transformations: Reflect over y axis

x	y
-1·-1	1/2
-1·0	1
-1·1	2

x	y
1	1/2
0	1
-1	2

f) Graph $f(x)=-3^x$ a=3



Domain: $(-\infty,\infty)$

Range: $(-\infty,0)$

Horizontal asymptote: y=0 or x axis

Key points and transformations: Reflect x axis

x	y
-1	1/3·-1
0	1·-1
1	3·-1

x	y
-1	-1/3
0	-1
1	-3

Graphing

9.4N – Logarithmic Functions

A. Inverses of exponential functions.

- The logarithmic function $y = \log_a x$ is the inverse of the exponential function $\underline{y = a^x}$.
- Domain $y = a^x$: $\underline{(-\infty, \infty)}$. Range $y = a^x$: $\underline{(0, \infty)}$.
- Domain $y = \log_a x$: $\underline{(0, \infty)}$. Range $y = \log_a x$: $\underline{(-\infty, \infty)}$

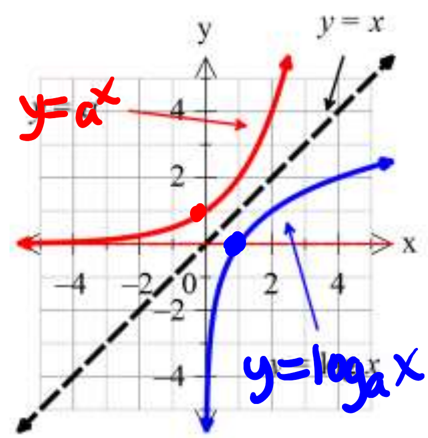
Domain of the logarithmic function =

$\underline{\text{Range}}$ of the exponential function $= (0, \infty)$

Range of the logarithmic function =

$\underline{\text{Domain}}$ of the exponential function $= (-\infty, \infty)$

$a > 1$



5^x

$5^1 = 5$

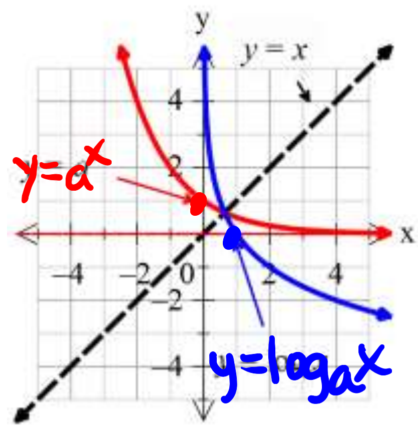
$5^0 = 1$

$5^{-1} = \frac{1}{5}$

$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

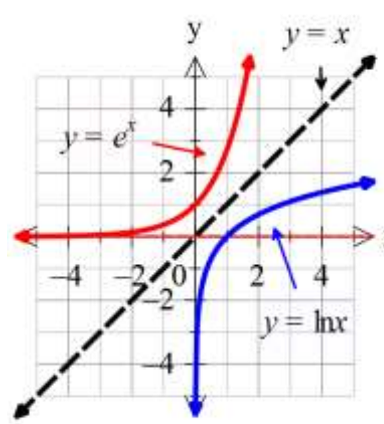
★ **Caution!** You can't take the log of zero or of a negative because it is impossible to get zero or a negative by raising a positive base to an exponent. **The argument of a logarithmic function must be greater than zero.**

$0 < a < 1$



Properties of the Logarithmic Function $f(x) = \log_a x$

- The x-intercept is $\underline{1}$. There is $\underline{\text{no}}$ y-intercept.
- The vertical asymptote of the graph is $\underline{x = 0}$.
- The logarithmic function is $\underline{\text{decreasing}}$ if $0 < a < 1$ and $\underline{\text{increasing}}$ if $a > 1$. The function is one-to-one.
- Since $y = \log_a x$ is the inverse of $y = a^x$ and the graph $y = a^x$ contains the points $\boxed{\left(-1, \frac{1}{a}\right), (0,1), \text{ and } (1, a)}$ keypoints then the graph of reversed $y = \log_a x$ contains the points $\left(\underline{\frac{1}{a}}, \underline{-1}\right), \left(\underline{1}, \underline{0}\right)$, and $\left(\underline{a}, \underline{1}\right)$.



Common Logarithmic Function: If the base of a logarithmic function is the number 10, then we have the common logarithmic function. If the base a of the logarithmic function is not indicated, it is understood to be 10. That is, $y = \log_{10} x$ if and only if $x = 10^y$.

Natural Logarithms: If the base of a logarithmic function is the number *e*, then we have the natural logarithm function (abbreviated ln). That is, $y = \ln x$ if and only if $x = e^y$.

- $y = \ln_e x$ is the $\underline{\text{inverse}}$ of $y = e \wedge x$

$\ast$ To find the domain set argument > 0

B. Finding the domain of logarithmic functions.

$\log_2 x$

1. $f(x) = \log_2(x+3)$

$x+3>0$
$-3 \quad -3$
$x>-3$
Domain $(-3,\infty)$

2. $h(x) = -\log_{\frac{1}{2}} x$

$x>0$
Domain $(0,\infty)$

3. $g(x) = \ln(-x-5)$

$-x-5>0$
$-x>5$
$\div -1 \quad \div -1$
$x<-5$
Domain $(-\infty,-5)$

C. Graphing logarithmic functions.

Steps for Graphing Logarithmic Functions:

exponential key points

x	y
-1	1/a
0	1
1	a

logarithmic key points

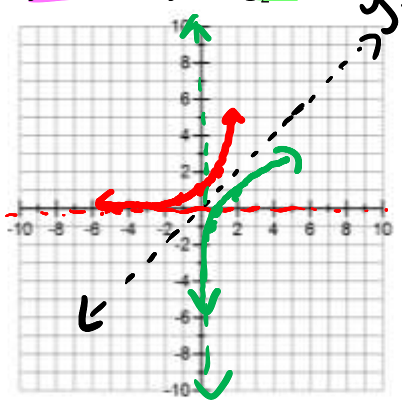
x	y
1/a	-1
1	0
a	1

1. Find the domain
2. Find the asymptotes
3. Graph the asymptotes
4. Find the 3 key points $(1,0)$, $(a,1)$, **and** $\left(\frac{1}{a},-1\right)$ and apply the appropriate transformations.
5. Plot your points and connect them to form a smooth curve.
6. Find the range

Examples: Graph the following functions.

a) $y=2^x$ and $y=\log_2 x$

x	y
-1	1/2
0	1
1	2



$y=x$

Domain: $(0,\infty)$

Asymptotes: $x=0$

Key points and transformations: no transf.

x	y
1/2	-1
1	0
2	1

$y=\ln e$

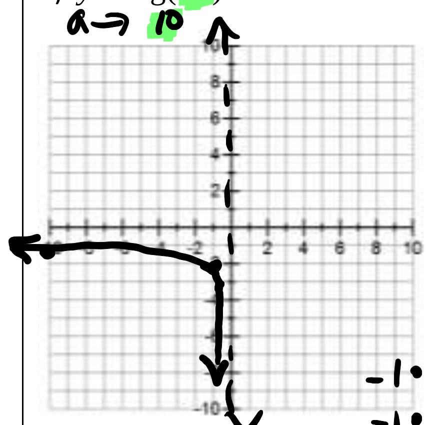
Range: $(-\infty,\infty)$

b) $y=\log(-x)-2$.

$y=\log_{10}(x)$

$a \to 10$

Domain
$-x>0$
$\div -1 \quad \div -1$
$x<0$



Domain: $(-\infty,0)$

Asymptotes: $x=0$

Key points and transformations:
① reflect over y axis
② Down 2

x	y
$-1\cdot 1/10$	$-1-2$
$-1\cdot 1$	$0-2$
$-1\cdot 10$	$1-2$

$$\boxed{\begin{array}{c|c} x & y \\ \hline -1/10 & -3 \\ -1 & -2 \\ -10 & -1 \end{array}}$$

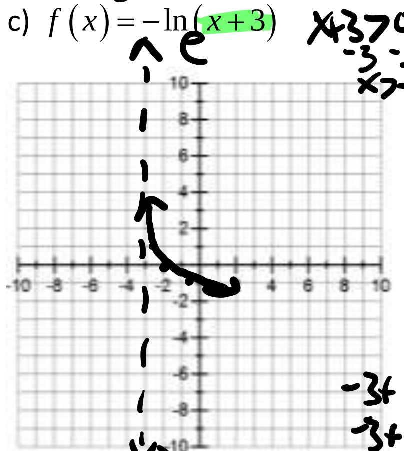
$\log_{10}(x)$

Range: $(-\infty,\infty)$

c) $f(x)=-\ln(x+3)$

e

$x+3>0$
$-3 \quad -3$
$x>-3$



Domain: $(-3,\infty)$

Asymptotes: $x=-3$

Key points and transformations:
① reflect over x axis
② left 3

x	y
$-3+\frac{1}{e}$	$-1\cdot -1$
$-3+1$	$0\cdot -1$
$-3+e$	$1\cdot -1$

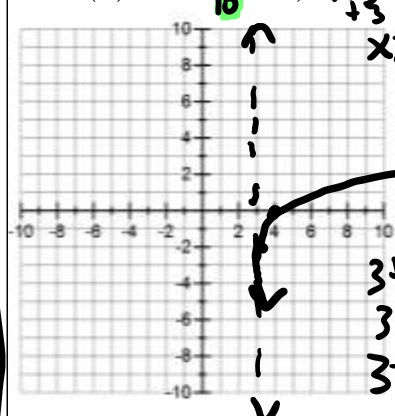
$$\boxed{\begin{array}{c|c} x & y \\ \hline -2.63 & 1 \\ -2 & 0 \\ -.28 & -1 \end{array}}$$

$-3+\left(\frac{1}{e}\right)=-2.63$
$-3+1=-2$
$-3+e=-.28$

Range: $(-\infty,\infty)$

d) $f(x)=2\log_{10}(x-3)$

$x-3>0$
$+3 \quad +3$
$x>3$



Domain: $(3,\infty)$

Asymptotes: $x=3$

Key points and transformations:
vert. stretch of 2
right 3

x	y
$3+1/10$	$-1\cdot 2$
$3+1$	$0\cdot 2$
$3+10$	$1\cdot 2$

$$\boxed{\begin{array}{c|c} x & y \\ \hline 3\,1/10 & -2 \\ 4 & 0 \\ 13 & 2 \end{array}}$$

Range: $(-\infty,\infty)$

9.5 Solving Logarithmic Equations

D. Finding the inverse of a logarithmic function.

- **$\log_2 x$ means "the exponent to which we raise 2 to get *x*."**
  Pronounced "the logarithm, base 2, of *x*" or "log, base 2, of *x*"

$2^y = x$

★LOGARITHMS ARE EXPONENTS!★

- ***Logarithm:*** $\log_b a$ means the ***exponent*** to which we raise b to get a.
  b is called the ***base*** of the logarithm (the number being raised to the exponent).
  a is called the ***argument*** of the logarithm (the number you get when you raise the base to the exponent).

The ***logarithmic function of base b***, where $b > 0$ and $b \neq 1$ is denoted by $y = \log_b x$ and is defined by

★ $\boxed{y = \log_b x \text{ if and only if } x = b^y.}$ ★

exponent

b = base
y = exponent
x = argument/answer

Example: Change each exponential expression to an equivalent expression involving a logarithm.

a) $5^{\boxed{x}} = 625$

$\log_5 625 = x$

b) $\boxed{x}^3 = 64$

answer, exp.

$\log_x 64 = 3$

base

c) $3^{\boxed{2}} = x$

$\log_3 x = 2$

Example: Change each logarithmic expression to an equivalent expression involving an exponent.

a) $\log_3 x = 5$

base, ans, exp

$3^5 = x$

b) $\log_e 5 = x$

$e^x = 5$

c) $\log_m 2 = n$

$m^n = 2$

E. Evaluating Logarithms

- Instead of "$\log_2 8 = x$," think, what power of 2 equals 8? Or 2 to what power equals 8?
  - $2^x = 8$ $\quad 2^x = 2^3$ $\quad x = 3$
  - The answer would be 3 because $2^3 = 8$.

① rewrite in exponential form
② solve by getting same base

Example: Find the exact value of each logarithm without using a calculator.

a) $\log_3 9 = x$

$3^x = 9$

$3^x = 3^2$

$\boxed{x = 2}$

b) $\log_2 32 = x$

$2^x = 32$

$2^x = 2^5$

$\boxed{x = 5}$

c) $\log_6 1 = x$

$6^0 = 1$ $\quad 5^0 = 1$

$6^x = 1$

$\boxed{x = 0}$

d) $\log_5 \frac{1}{125} = x$

$5^x = \frac{1}{125}$

$5^x = \frac{1}{5^2}$

$5^x = 5^{-2}$

$\boxed{x = -2}$

e) $\log_7 \sqrt{7} = x$

$7^x = \sqrt[2]{7}$

$7^x = 7^{1/2}$

$\boxed{x = 1/2}$

use calculator to evaluate. Round answers to nearest ten thousandth.

a) $\log(5.83)$
.7657

b) $\log(-23)$
argument can't be negative
no solution

c) $\ln(21.4)$
3.0634

d) $\frac{\ln(6)}{2}$
.8959

$\log_{10}$ ← base 10
$\ln_e$ ← base e

9.5 N – Solving Logarithmic Equations

A. Review

Change each logarithmic statement into an equivalent exponential statement.

1. $\log_8 64 = 2$ $\quad 8^2 = 64$
2. $\log_2 \frac{1}{16} = -4$ $\quad 2^{-4} = \frac{1}{16}$
3. $\log_{10} 8 = x$ $\quad 10^x = 8$
4. $\ln_e x = 5$ $\quad e^5 = x$

Change each exponential statement into an equivalent logarithmic statement.

1. $\boxed{4}^x = 27$ $\quad \log_4 27 = x$
2. $\boxed{3}^{-4} = \frac{1}{81}$ $\quad \log_3 \frac{1}{81} = -4$
3. $\boxed{9}^x = 32$ $\quad \log_9 32 = x$

Solve the following equation using the laws of exponents. solve by getting same base

1. $16^{m+2} = 64$
$4^{2(m+2)} = 4^3$
$2(m+2) = 3$
$2m+4 = 3$
$2m = -1$
$\boxed{m = -1/2}$

2. $9^{-3n} = 243$
$3^{2(-3n)} = 3^5$
$2(-3n) = 5$
$\frac{-6n}{-6} = \frac{5}{-6}$
$\boxed{n = -\frac{5}{6}}$

B. Solving Logarithmic and Exponential Equations

- Use the properties of logarithms and exponents to manipulate the equations.
    a. Remember the *exponential* property: $a^u = a^v \Leftrightarrow u = v$.
- Try rewriting as an *exponential* function: $y = \log_a x \Leftrightarrow x = a^y$ **or** as a *logarithmic* equation: $x = a^y \Leftrightarrow y = \log_a x$

* If it is a log rewrite as exponential.
* If it is an exponential, rewrite as a log.

Rewrite as exponential | Rewrite as log

Examples:

a) $\log_{18} 324 = x$
$18^x = 324$
$18^x = 18^2$
$\boxed{x = 2}$

b) $\boxed{6}^{x-4} = 11$ exp.
$\log_6 11 = x - 4$
$+4 \qquad +4$
$\boxed{x = \log_6(11) + 4}$ or $\boxed{x = 4 + \log_6 11}$

c) $\ln e^{2x} = 6$
$e^6 = e^{2x}$
$\frac{6}{2} = \frac{2x}{2}$
$\boxed{x = 3}$

d) $\frac{3 \cdot (10)^{3-x}}{3} = \frac{7}{3}$
$10^{3-x} = \frac{7}{3}$
$\log_{10}\left(\frac{7}{3}\right) = 3 - x$
$-3 \qquad -3$
$\frac{-3 + \log_{10}\left(\frac{7}{3}\right)}{-1} = \frac{-x}{-1}$
$\boxed{x = 3 - \log_{10}(7/3)}$ or $x = -\log_{10}(7/3) + 3$

e) $\log_3(3x-1) = 2$
$3^2 = 3x - 1$
$9 = 3x - 1$
$+1 \qquad +1$
$\frac{10}{3} = \frac{3x}{3}$
$\boxed{x = 10/3}$

f) $\boxed{2}^{-x} = 1.5$
$\frac{1 \log_2 1.5}{-1} = \frac{-x}{-1}$
$\boxed{x = -\log_2(1.5)}$

g) $\log_6 216 = 3x + 2$ exp
$6^{3x+2} = 216$
$6^{3x+2} = 6^3$
$3x + 2 = 3$
$-2 \qquad -2$
$3x = 1$
$\boxed{x = \frac{1}{3}}$

h) $\boxed{e}^{4x+3} = 9$
$\ln_e 9 = 4x + 3$
$-3 \qquad -3$
$\frac{-3 + \ln_e(9)}{4} = \frac{4x}{4}$
$\boxed{x = \frac{-3 + \ln_e 9}{4}}$

9.6 – Properties of Logarithms

inverse operations cancel

$\sqrt{x^2} = x$

- ***Remember:*** *Definition of Logarithm*: $y = \log_a x \Leftrightarrow a^y = x$

$\log_a x = y$ exp $\quad a^y = x$

A. Properties of Logarithms

For any positive numbers M, N, and a, where $a \neq 1$ and r is any real number:

cancellation laws

If $a^0 = 1$ then	If $a^1 = a$ then	If $a^M = \log_a M$ then	If $a^r = a^r$ then
▪ $\log_a 1 = \underline{0}$	▪ $\log_a a = \underline{1}$	▪ $a^{\log_a M} = \underline{M}$	▪ $\log_a a^r = \underline{r}$

- ▪ $\log_a (MN) = \log_a M + \log_a N$
- ▪ $\log_a M^r = r \cdot \log_a M$
- ▪ $\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$
- ▪ $\log_a M = \log_a N \Leftrightarrow M = N$

Change of Base Formula:

▪ $\log_a M = \frac{\log_b M}{\log_b a}$ ▪ $\log_a M = \frac{\log M}{\log a}$ (common log) ▪ $\log_a M = \frac{\ln M}{\ln a}$ (natural log)

NOTATION

$\log_e(x) = \ln(x)$

$\log(x) = \log_{10}(x)$

Examples: Find the exact value of each expression. (Do not use a calculator).

a) $\log_4 1 = 0$

b) $5^{\log_5 3} = 3$

c) $\log_7 7^{-1} = -1$

d) $\ln e^1 = 1$

e) $\log_2 64 \quad \log_2 2^6 = 6$

f) $\log_7 \frac{1}{\sqrt[3]{49}} = \log_7 \frac{1}{49^{1/3}} = \log_7 \frac{1}{(7^2)^{1/3}} = \log_7 \frac{1}{7^{2/3}} = \log_7 7^{-2/3} = \boxed{-2/3}$

Expand

Examples: Write each expression as a *sum/difference* of logarithms. Express powers as factors.

a) $\log 5x = \log 5 + \log x$

b) $\ln \frac{3}{x} = \ln 3 - \ln x$

c) $\log_7 (x^5)$

$5 \log_7 x$

d) $\ln(x^2 e^x)$

$\ln x^2 + \ln e^x$

$2\ln x + x \ln e$

e) $\log \frac{\sqrt[4]{x}}{\sqrt[4]{y}}$

$\log \frac{x^{1/4}}{y^{1/4}}$

$\log x^{1/4} - \log y^{1/4}$

$1/4 \log x - 1/4 \log y$

f) $\ln \frac{y^4}{x^5}$

$\ln y^4 - \ln x^5$

$4 \ln y - 5 \ln x$

Examples: Write each expression as a single logarithm.

Condense

a) $\ln 8+\ln x$

$\ln(8\cdot x)$

$\ln(8x)$

b) $\log u-\log v$

$\log\left(\frac{u}{v}\right)$

c) $\frac{1}{4}\log x$

$\log x^{1/4}$

d) $\log_7 u+3\log_7 v$

$\log_7 u+\log_7 v^3$

$\log_7(u\cdot v^3)$

e) $4\ln(uv)-3\ln(vw)$

$\ln(uv)^4-\ln(vw)^3$

$\boxed{\ln\frac{(uv)^4}{(vw)^3}}$ simplify $\ln\frac{u^4v^{4-3}}{v^3w^3}\rightarrow \ln uv$

f) $\log(x-4)+\log(6x+5)$ FOIL

$\log(x-4)(6x+5)$

$\log 6x^2+5x-24x-20$

$\boxed{\log 6x^2-19x-20}$

Examples: Use the *change of base* formula to evaluate each logarithm. 4 dec. places

a) $\log_6 9$

$\frac{\log 9}{\log 6}=1.2263$

b) $\log_{\sqrt{2}} 7=\frac{\log 7}{\log\sqrt{2}}$

$=5.6147$

c) $\log_\pi \sqrt{3}=\frac{\log\sqrt{3}}{\log\pi}$

$=.4799$

Examples: Write the expression using only *natural logarithms*. ln use change of base

a) $\log_7 30=\frac{\ln 30}{\ln 7}=1.7479$

b) $\log_4 10=\frac{\ln 10}{\ln 4}=1.6610$

Examples: Write the expression using only *common logarithms*. log use change of base

a) $\log_6 y=\frac{\log y}{\log 6}$

b) $\log_2(d+e)=\frac{\log(d+e)}{\log 2}$

Examples: Use properties of logarithms to find the exact value of each expression. (Do not use a calculator).

condense

a) $\log_7 21-\log_7 3$ simplify

$\log_7\left(\frac{21}{3}\right)$ cancellation log

$\log_7 7^1$

$\boxed{1}$

b) $5^{\log_5 6+\log_5 7}$ condense

$5^{\log_5(6\cdot 7)}$

cancellation

$(6\cdot 7)$

$\boxed{42}$

c) $\log_4 11\cdot\log_{11} 256$

bases not same

can't expand

can't do

9.7 N - Solving Logarithmic Equations

Solve by getting same base.

A. Review

1) $\log_3 x = 4$

$3^4 = x$

$\boxed{81 = x}$

2) $\frac{27}{27}\left(\frac{1}{3}\right)^{x/5} = \frac{3}{27}$

$\left(\frac{1}{3}\right)^{x/5} = \frac{1}{9}$

$\left(\frac{1}{3^1}\right)^{x/5} = \frac{1}{3^2}$

$\left(3^{-1}\right)^{x/5} = 3^{-2}$

$3^{-x/5} = 3^{-2}$

$5 \cdot -\frac{x}{5} = -2 \cdot 5$

$\frac{-x}{-1} = \frac{-10}{-1}$

$\boxed{x = 10}$

3) $\frac{16 \cdot 4^{x/3}}{16} = \frac{1024}{16}$

$4^{x/3} = 64$

$4^{x/3} = 4^3$

$3 \cdot \frac{x}{3} = 3 \cdot 3$

$\boxed{x = 9}$

B. Use the Properties of Logarithms and Exponents to solve equations.

check for extraneous solutions if it is a log.

Tips

- To avoid extraneous solutions, determine the domain of the variable first.
- Use the *properties of logarithms and exponents* to manipulate the equations.
- Try rewriting as an exponential or logarithmic function: $y = \log_a x \Leftrightarrow x = a^y$
- Remember the properties: $\log_a M = \log_a N \Leftrightarrow M = N$ and $a^u = a^v \Leftrightarrow u = v$ (Make the bases the same).
- Check your solution by substituting into the original equation.

* If it is an exponential rewrite as a log and solve.

a) $4^x = 37$

$\log_4 37 = x$

use change of base to plug in calc.

$x = \frac{\log 37}{\log 4}$ ← change of base

$\boxed{x \approx 2.6047}$

b) $2.05^x = 4.36$

$\log_{2.05} 4.36 = x$

change of base in calc.

$x = \frac{\log 4.36}{\log 2.05}$

$\boxed{x \approx 2.0513}$

c) $\frac{30e^{0.014x}}{30} = \frac{600}{30}$

$e^{.014x} = 20$

$\ln_e 20 = .014x$

same as

$\frac{\ln 20}{.014} = \frac{.014x}{.014}$

$\boxed{x \approx 213.9809}$

d) $8 - 5e^{-x} = -12$

$-8 \qquad -8$

$\frac{-5e^{-x}}{-5} = \frac{-20}{-5}$

$e^{-x} = 4$

$\ln_e 4 = -x$

same as

$\frac{\ln 4}{-1} = \frac{-x}{-1}$

$x = -\ln 4 \approx \boxed{-1.3863}$

e) $2^{4-x} - 7 = 14$

$+7 \quad +7$

$2^{4-x} = 21$

$\log_2 21 = 4 - x$

$-4 \quad -4$

$\frac{\log_2(21) - 4}{-1} = \frac{-x}{-1}$ ↙ change of base form.

$x = -\log_2(21) + 4$

$x \approx -\frac{\log 21}{\log 2} + 4 \approx \boxed{-.3923}$

f) $\log_4 x = \log_4(3x - 8)$

* check your answer

$x = 3x - 8$

$-x \qquad -x$

$\frac{8}{2} = \frac{2x}{2}$

$\boxed{x = 4}$

g) $\ln x^2 = 8$ * check ans.

$$\sqrt[2]{e^8} = \sqrt{\cancel{x^2}}$$

$$\pm e^4 = x$$

$$\boxed{\pm 54.5982 \approx x}$$

skip ~~i) $2\log_4 x - \log_4 9$~~

condense to 1 log

k) $\ln(5x) - \ln(10) = 5$

$$\ln_e\left(\frac{5x}{10}\right) = 5$$

rewrite as exp

$$e^5 = \frac{\cancel{5}x}{\cancel{10}\,2}$$

$$2 \cdot e^5 = \frac{x}{\cancel{2}} \cdot 2$$

$$x = 2e^5$$

$$\boxed{x \approx 296.8263}$$

m) [redacted]

condense

$$\log_2 4 + \log_2 9 = \log_2(5x-4)$$

$$\log_2(4 \cdot 9) = \log_2(5x-4)$$

$$\log_2(36) = \log_2(5x-4)$$

$$36 = 5x - 4$$

$$\frac{40}{5} = \frac{5x}{5}$$

$$\boxed{8 = x}$$

check answer

* If log, try rewriting as an exp. and solve.

h) $-4\log(x+5) - 3 = -4$

$$+3 \quad +3$$

$$\frac{-4\log(x+5)}{-4} = \frac{-1}{-4}$$

$$\log_{10}(x+5) = \frac{1}{4}$$

* check answer

$$10^{\frac{1}{4}} = x + \cancel{5}$$

$$-5 \quad \cancel{-5}$$

$$\boxed{x \approx -3.2217}$$

condense

j) [redacted]$\log_2(x-1) + \log_2 4 = 5$

$$\log_2(4(x-1)) = 5$$

$$2^5 = 4(x-1)$$

*check answer

$$32 = 4x - 4$$

$$36 = 4x$$

$$\boxed{x = 9}$$

check answer

l) [redacted]

$$\log_6 4 + \log_6(x+3) = \log_6 x$$

$$\log_6(4(x+3)) = \log_6 x$$

$$4(x+3) = x$$

$$4x + 12 = x$$

$$-x \quad -12 \quad -x \quad -12$$

$$\frac{3x}{3} = \frac{-12}{3}$$

$$\cancel{x = -4}$$

$\boxed{\text{extraneous}}$

n) condense

$$\log_5 20 - \log_5 2 = \log_5 5x$$

$$\log_5\left(\frac{20}{2}\right) = \log_5(5x)$$

$$\log_5(10) = \log_5(5x)$$

$$\frac{10}{5} = \frac{5x}{5}$$

$$\boxed{2 = x}$$

check answer

9.8N - Financial Models & Exp. Growth & Decay Models

A. Review

Express each percent as a decimal.

1) 3% 2) 13.5% 3) 102%

B. ***Simple Interest***: If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is $\boldsymbol{I = Prt}$.

Example1: What is the interest due if $1000 is borrowed for 9 months at a simple interest rate of 5% per year?

Example2: If you borrow $7000 and, after 6 months pay off the loan in the amount of $7,500, what yearly rate of interest was charged?

C. ***Compound Interest:*** When the interest due at the end of a payment period is added to the principal so that the interest computed at the end of the next payment period is based on the new principal amount (old principal plus interest), the interest is said to have been ***compounded***. ***Compound interest*** is interest paid on principal and previously earned interest.

The amount A after t years due to a principal P invested at an annual interest rate t compounded n times per year is $\boldsymbol{A = P\cdot\left(1+\frac{r}{n}\right)^{nt}}$. *Present value* or to find the principal: $\boldsymbol{P = A\cdot\left(1+\frac{r}{n}\right)^{-nt}}$.

Example 1: Investing $1000 at an annual rate of 9% compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 1 year:

Annual Compounding (n = 1):

Semiannual Compounding (n = 2):

Quarterly Compounding (n = 4):

Monthly Compounding (n = 12):

Daily Compounding (n = 365):

Example 2: How much money must be invested now in order to end up with $20,000 in 10 years at 5% compounded quarterly?

D. ***Continuous Compounding***

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is $\boldsymbol{A = Pe^{rt}}$. *Present value* or to find the Principal: $\boldsymbol{P = Ae^{-rt}}$.

Example 1: Find the amount A that results from investing a principle P of $1000 at an annual rate r of 9% compounded continuously for a time t of 1 year.

Example 2: How much money must be invested now in order to end up with $20,000 in 10 years at 3.8% compounded continuously?

E. Exponential Growth and Decay Models

***Law of Uninhibited Growth or Decay*:**
Many natural phenomena have been found to follow the law that an amount A varies with time t according to the function $A(t) = A_0 e^{kt}$, where A_0 is the original amount at time $t = 0$ and k is a constant of growth or decay (growth if $k > 0$, decay if $k < 0$.)

Example 1: The number N of bacteria present in a culture at time t hours obeys the law of uninhibited growth where $N(t) = 1000e^{0.01t}$.

a) Determine the number of bacteria at $t = 0$ hours.

b) What is the growth rate of the bacteria?

c) What will the population be after 4 hours?

d) When will the number of bacteria reach 1700?

e) When will the number of bacteria double?

Example 2: Iodine 131 is a radioactive material that decays according to the function $A(t) = A_0 e^{-0.087t}$, where A_0 is the initial amount present and A is the amount present at time t (in days). Assume that a scientist has a sample of 100 grams of iodine 131.

a) What is the decay rate of iodine 131?

b) How much iodine 131 is left after 9 days?

c) When will 70 grams of iodine 131 be left?

d) What is the half-life of iodine 131? (when $A = \frac{1}{2} A_0$.)