SM3 9.8 Financial Models and Exponential Growth and Decay Notes

When the interest due at the end of a payment period is added to the principal so that the interest computed at the end of the next payment period is based on the new principal amount (old principal plus interest), the interest is said to have been ***compounded***. ***Compound interest*** is interest paid on principal and previously earned interest.

A = Final Amount
P = original balance
r = interest rate (as a decimal)
n = # of times interest applied per yr
t = time

Compound Interest Formula

The amount A after t years due to a principal P invested at an annual interest rate t compounded n times per year is $\boldsymbol{A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}}$.

Example: Investing $1000 at an annual rate of 9% (or 9% per annum) compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 1 year:

P = 1000 r = .09 t = 1

Annual Compounding ($n = 1$): $A = 1000\left(1 + \frac{.09}{1}\right)^{1 \cdot 1} = \$1090.00$

Semiannual Compounding ($n = 2$): $A = 1000\left(1 + \frac{.09}{2}\right)^{2 \cdot 1} = \$1092.03$

Quarterly Compounding ($n = 4$): $A = 1000\left(1 + \frac{.09}{4}\right)^{4 \cdot 1} = \$1093.08$

Monthly Compounding ($n = 12$): $A = 1000\left(1 + \frac{.09}{12}\right)^{12 \cdot 1} = \$1093.81$

Daily Compounding ($n = 365$): $A = 1000\left(1 + \frac{.09}{365}\right)^{365 \cdot 1} = \$1094.16$

Continuous Compounding

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is $\boldsymbol{A = Pe^{rt}}$.

Example: Find the amount A that results from investing a principle P of $1000 at an annual rate r of 9% compounded continuously for a time t of 1 year.

$A = 1000e^{.09(1)}$

$A = 1000e^{.09} = \$1094.17$

***Present Value*:** The amount of money that must be invested now in order to end up with a given amount after a certain amount of time.

Use $A = P \cdot \left(1+\frac{r}{n}\right)^{nt}$ and $A = Pe^{rt}$ from the above examples. Substitute the numbers in and the solve for *P*.

Example: How much money must be invested now in order to end up with \$20,000 in 10 years at

A t

a) 5% compounded quarterly?

r n=4

$r = .05$

$$\frac{20000}{\left(1+\frac{.05}{4}\right)^{40}} = \frac{P\cdot\cancel{\left(1+\frac{.05}{4}\right)^{4\cdot 10}}}{\cancel{\left(1+\frac{.05}{4}\right)^{40}}}$$

$$\boxed{P = 12,168.27}$$

b) 3.8% compounded continuously?

$r = .038$

$A = Pe^{rt}$

$$\frac{20000}{e^{.38}} = \frac{P_o \cancel{e^{.38}}}{\cancel{e^{.38}}} \quad .038(10) = .38$$

$$\boxed{P = 13,677.23}$$

Exponential Growth and Decay Models

***Law of Uninhibited Growth or Decay*:**

Many natural phenomena have been found to follow the law that an amount A varies with time t according to the function $A(t) = A_0 e^{kt}$, where A_0 is the original amount at time $t = 0$ and k is a constant of growth or decay (growth if $k > 0$, decay if $k < 0$.)

Example: The number N of bacteria present in a culture at time t hours obeys the law of uninhibited growth where $N(t) = 1000e^{0.01t}$.

a) Determine the number of bacteria at $t = 0$ hours. 1000

b) What is the growth rate of the bacteria? .01 or 1%

c) What will the population be after 4 hours? $t = 4$

$$N(4) = 1000e^{.01(4)} = 1000e^{(.01\cdot 4)} \quad (.04)$$

$$N(4) \approx 1040 \text{ bacteria}$$

d) When will the number of bacteria reach 1700?

*rewrite as log

what time?

$$\frac{1700}{1000} = \frac{\cancel{1000}e^{.01t}}{\cancel{1000}}$$

$1.7 = e^{.01t}$ same as $\boxed{e^{.01t} = 1.7}$

$$\frac{\ln 1.7}{.01} = \frac{.01t}{.01}$$

$$\boxed{t \approx 53.1 \text{ hours}}$$

e) When will the number of bacteria double?

*rewrite as log

$$\frac{2000}{1000} = \frac{\cancel{1000}e^{.01t}}{\cancel{1000}}$$

$2 = e^{.01t}$

$$\boxed{e^{.01t} = 2}$$

$$\frac{\ln 2}{.01} = \frac{.01t}{.01}$$

$$\boxed{t \approx 69.3 \text{ hours}}$$

Example: Iodine 131 is a radioactive material that decays according to the function $A(t)=A_0e^{-0.087t}$, where A_0 is the initial amount present and A is the amount present at time t (in days). Assume that a scientist has a sample of 100 grams of iodine 131.

A_0

a) What is the decay rate of iodine 131?

$-.087$ or -8.7%

b) How much iodine 131 is left after 9 days?

$A(9) = 100e^{\cancel{-.087t}\,-.783}$ ≈ 45.7 grams

c) When will 70 grams of iodine 131 be left?

*rewrite as log

$\frac{70}{100} = \frac{100e^{-.087t}}{100}$

$\Rightarrow .7 = e^{-.087t}$

$\boxed{e^{-.087t} = .7}$

$\frac{\ln .7}{-.087} = \frac{-.087t}{-.087}$

$\boxed{t = 4.1 \text{ days}}$

d) What is the half-life of iodine 131? (when $A=\frac{1}{2}A_0$.)

*rewrite as log

$\frac{50}{100} = \frac{100e^{-.087t}}{100}$

$.5 = e^{-.087t}$

$\boxed{e^{-.087t} = .5}$

$\frac{\ln(.5)}{-.087} = \frac{-.087t}{-.087}$

$\boxed{t = 7.9 \text{ days}}$

Example: Chemists define the acidity or alkalinity of a substance according to the formula

$pH = -\log H^+$, where H^+ is the hydrogen ion concentration, measured in moles per liter. Solutions with a pH value of less than 7 are acidic. Solutions with a pH value of greater than 7 are basic. Solutions with a pH of 7 (such as pure water) are neutral.

a) Suppose you test apple juice and find that the hydrogen ion concentration is $H^+ = 0.0003$. Find the pH value and determine whether the juice is basic or acidic.

$pH = -\log(.0003) = 3.5$

Acidic

b) Suppose you test ammonia and find that the hydrogen ion concentration is $H^+ = 1.3 \times 10^{-9}$. Find the pH value and determine whether the ammonia is basic or acidic.

$Ph = -\log(1.3 \times 10^{-9}) \approx 8.9$

basic