3.5 Solving Quadratic Equations by Factoring

Zero Product Property: If the product of several factors is equal to zero, then at least one of the factors is equal to zero.

- The only way to end up with zero when you multiply is if one of the numbers being multiplied is zero.
- If a and b are real numbers and $a \cdot b = 0$, then $a = 0$ or $b = 0$ or both.

★ **This is only true if one side of the equation is zero.**

If $a \cdot b = 1$, it ***does not mean*** that $a = 1$ or $b = 1$. $(2)(\frac{1}{2}) = 1$, $(\frac{3}{4})(\frac{4}{3}) = 1$, etc.

DON'T split up $(x+5)(x-3) = 1$ into $x + 5 = 1$ and $x - 3 = 1$. **That's wrong!**

Solving Quadratic Equations by Factoring:

1. Get a zero on one side of the equation.
2. Factor completely.
3. Set each factor *containing a variable* equal to 0.
4. Solve the resulting equations.

Examples: Solve each equation by factoring.

a) $(x-3)(x+5) = 0$

$x - 3 = 0$
$x + 5 = 0$
$\boxed{x = 3 \quad x = -5}$

b) $3x(x+4) = 0$

$3x = 0$
$x + 4 = 0$
$\boxed{x = 0 \quad x = -4}$

c) $2(x+5)(3x-4) = 0$

$x + 5 = 0$
$3x - 4 = 0$
$+4 \quad +4$
$\frac{3x}{3} = \frac{4}{3}$
$\boxed{x = -5 \quad x = 4/3}$

d) $(x+7)^2 = 0$

$x + 7 = 0$
$\boxed{x = -7}$

e) $x^2 + 7x + 6 = 0$

$(x+6)(x+1) = 0$
$\boxed{x = -6, x = -1}$

f) $x^2 + 21 = 10x$

$x^2 - 10x + 21 = 0$
$(x-7)(x-3) = 0$
$\boxed{x = 7, x = 3}$

g) $-x^2 - 10x = 25$

$-x^2 - 10x - 25 = 0$
$\frac{-1(x^2 + 10x + 25)}{-1} = \frac{0}{-1}$
$(x+5)^2 = 0$
$\boxed{x = -5}$

h) $x^2 - 36 = 0$

$(x+6)(x-6) = 0$
$\boxed{x = -6, x = 6}$

i) $-2x^2 + 14x = 24$

$-2x^2 + 14x - 24 = 0$
$-2(x^2 - 7x + 12) = 0$
$-2(x-4)(x-3) = 0$
$\boxed{x = 4, x = 3}$

j) $12x^2 - 18x = 12$

$12x^2 - 18x - 12 = 0$
$6(2x^2 - 3x - 2) = 0$
-4
$-4 \cdot 1$
$2x^2 - 4x + x - 2 = 0$
$2x(x-2) + 1(x-2) = 0$
$(2x+1)(x-2) = 0$
$\boxed{x = -1/2, x = 2}$

Tips for solving story problems:

- Identify what you know.
- What are you trying to find out?
- Draw a picture or diagram to help you visualize the situation.
- Carefully define your variables.
- Translate the words into symbols.
- Use appropriate units.
- Make sure your answer makes sense.

Hints:

- ***Sum:*** + ***Difference:*** – ***Product:*** × ***Quotient:*** ÷
- ***Consecutive Integers:*** $1\text{st}\ \# = x,\ \ 2\text{nd}\ \# = x+1,\ \ 3\text{rd}\ \# = x+2,$ etc.
- ***Consecutive even integers*** or ***consecutive odd integers:***
  $1\text{st}\ \# = x,\ \ 2\text{nd}\ \# = x+2,\ \ 3\text{rd}\ \# = x+4,$ etc.
- ***Area of a rectangle:*** Area = length × width.
- ***Perimeter of a rectangle:*** Add the lengths of its sides together.
- ***Dimensions:*** length and width

Examples:

1. Find two consecutive integers whose product is 210.

$x(x+1)=210$
$x^2+x-210=0$
$(x+15)(x-14)=0$
$x=-15,\ x=14$

$14\cdot 15$

$\boxed{\text{14 and 15 or } -15 \text{ and } -14}$

2. Find two consecutive odd integers whose product is 35.

$x(x+2)=35$
$x^2+2x-35=0$
$(x+7)(x-5)=0$
$x=-7,\ x=5$

$\boxed{\text{5 and 7 or } -7 \text{ and } -5}$

3. Find two consecutive even integers whose product is 120.

$x(x+2)=120$
$x^2+2x-120=0$
$(x+12)(x-10)=0$
$x=-12,\ x=10$

$\boxed{\text{10 and 12 or } -12 \text{ and } -10}$

4. The product of two numbers is -24. The second number is 18 more than three times the first number. What are the two numbers?

x
$3x+18$

$x(3x+18)=-24$
$3x^2+18x+24=0$
$3(x^2+6x+8)=0$
$3(x+4)(x+2)=0$
$x=-4,\ x=-2$

-4
$3(-4)+18=6$

-2
$3(-2)+18=12$

$\boxed{-4 \text{ and } 6 \quad \text{or} \quad -2 \text{ and } 12}$

5. An envelope is 3 inches longer than it is wide. The area is 180 in^2. What are the dimensions of the envelope?

$x(x+3)=180$
$x^2+3x-180=0$
$-12\cdot 15$
$(x-12)(x+15)=0$
$\boxed{x=12},\ \cancel{x=-15}$
*length & width can't be negative

$\boxed{12 \text{ inches} \times 15 \text{ inches}}$

6. A rock is thrown upward off the top of a cliff. Its height in feet above the ground after *t* seconds is given by the function $h(t)=-16t^2+48.$

a. What is the height of the cliff? (In other words, how high is the rock at $t=0$?)

$h(0)=-16(0)^2+48$
$=\boxed{48 \text{ feet}}$

b. How high is the rock after 1 second?

$h(1)=-16(1)^2+48$
$=-16+48=\boxed{32 \text{ feet}}$

c. How long does it take for the rock to hit the ground? (hint: when the rock hits the ground the height will be 0 so h(t)=0)

$0=-16t^2+48$
$-48 \qquad -48$

$\frac{-48}{-16}=\frac{-16t^2}{-16}$
$\sqrt{3}=\sqrt{t^2}$

$\boxed{t=1.73 \text{ seconds}}$

3.6 The Square Root Principle

Example: How many numbers can be squared to get 9? In other words, how many solutions are there to the equation $x^2 = 9$? What are they? What about the equation $x^2 = -9$?

3, −3

3i, −3i

All numbers except zero have two square roots, a positive square root and a negative square root.

The $\sqrt{\ }$ symbol means the positive square root. Both roots must be considered when solving an equation by taking square roots, so we use the $\pm$ symbol to include both roots.

- ***The number i:*** i is the number whose square is -1. That is, $i = \sqrt{-1}$ and $i^2 = -1$.
- ***Imaginary Number:*** A number that can be written in the form $a + bi$, where a and b are real numbers and $b \neq 0$. **Any number with an i in it is imaginary.**
- ***Square Root Property:*** If b is a real number and if $a^2 = b$, then $a = \pm\sqrt{b}$.
- ***Radicand:*** The number under the radical sign.

Solving Equations by Taking Square Roots: Do this when the equation has a perfect square and no other variables.

1. Get the perfect square alone on one side of the equation.
2. Use the square root property.
3. Simplify all square roots. Write the square roots of negative numbers in terms of *i*.
4. Solve for the variable, if necessary.

Examples: Solve each equation using the square root property. Include both real and imaginary solutions. Write your solutions in simplest radical form. Write imaginary solutions in the form $a + bi$.

a) $\sqrt{x^2} = \sqrt{50}$

$x = \pm\sqrt{50}$

$10 \cdot 5$

$2 \cdot 5$

$\boxed{x = \pm 5\sqrt{2}}$

b) $\frac{3d^2}{3} = \frac{48}{3}$

$\sqrt{d^2} = \sqrt{16}$

$d = \pm\sqrt{16}$

$\boxed{d = \pm 4}$

c) $\frac{2z^2}{2} = \frac{-48}{2}$

$\sqrt{z^2} = \sqrt{-24}$

$z = \pm i\sqrt{24}$

$12 \cdot 2$

$6 \cdot 2$

$2 \cdot 3$

$\boxed{z = \pm 2i\sqrt{6}}$

d) $p^2 - 3 = 42$

$+3 \quad +3$

$\sqrt{p^2} = \sqrt{45}$

$p = \pm\sqrt{45}$

$9 \cdot 5$

$3 \cdot 3$

$\boxed{p = \pm 3\sqrt{5}}$

e) $2n^2+\cancel{4}=104$
$\cancel{-4}\quad -4$

$\dfrac{\cancel{2}n^2}{\cancel{2}}=\dfrac{100}{2}$

$\sqrt{n^2}=\sqrt{50}$

$n=\pm\sqrt{50}$

$\boxed{n=\pm 5\sqrt{2}}$

f) $3q^2+\cancel{6}=-48$
$\cancel{-6}\quad -6$

$\dfrac{\cancel{3}q^2}{\cancel{3}}=\dfrac{-54}{3}$

$\sqrt{q^2}=\sqrt{-18}$

$q=\pm i\sqrt{18}$

$3\ 3\ 2$

$\boxed{q=\pm 3i\sqrt{2}}$

g) $\sqrt{16}=\sqrt{(y+1)^2}$

$\pm\sqrt{16}=y+1$

$\pm 4=y\cancel{+1}$
$-1\qquad \cancel{-1}$

$-1\pm 4=y$

$\boxed{3,-5=y}$

h) $\sqrt{(2m-5)^2}=\sqrt{-25}$

$2m\cancel{-5}=\pm 5i$
$\cancel{+5}\quad +5$

$\dfrac{\cancel{2}m}{\cancel{2}}=\dfrac{5\pm 5i}{2}$

$\boxed{m=\dfrac{5\pm 5i}{2}}$

i) $(r+4)^2\cancel{-10}=26$
$\cancel{+10}\ +10$

$\sqrt{(r+4)^2}=\sqrt{36}$

$r\cancel{+4}=\pm 6$
$\cancel{-4}\quad -4$

$r=-4\pm 6$

$\boxed{r=2,-10}$

j) $2\cdot -10=\cancel{2}\cdot\dfrac{1}{\cancel{2}}(n-7)^2$

$\sqrt{-20}=\sqrt{(n-7)^2}$

$\pm i\sqrt{20}=n-7$

$2\ 10$

$2\ 5$

$\pm 2i\sqrt{5}=n\cancel{-7}$
$+7\qquad \cancel{+7}$

$\boxed{n=7\pm 2i\sqrt{5}}$

k) $\dfrac{\cancel{5}(x+10)^2}{\cancel{5}}=\dfrac{0}{5}$

$\sqrt{(x+10)^2}=\sqrt{0}$

$x\cancel{+10}=\pm 0$
$\cancel{-10}\quad -10$

$\boxed{x=-10}$

l) $-4(w+3)^2+\cancel{6}=86$
$\cancel{-6}\ -6$

$\dfrac{\cancel{-4}(w+3)^2}{\cancel{-4}}=\dfrac{80}{-4}$

$\sqrt{(w+3)^2}=\sqrt{-20}$

$w+3=\pm i\sqrt{20}$

$2\ 10$

$\cancel{2}\ 5$

$w\cancel{+3}=\pm 2i\sqrt{5}$
$\cancel{-3}\quad -3$

$\boxed{w=-3\pm 2i\sqrt{5}}$

m) $\dfrac{\cancel{-2}(x-3)^2}{\cancel{-2}}=\dfrac{-32}{-2}$

$\sqrt{(x-3)^2}=\sqrt{16}$

$x\cancel{+3}=\pm 4$
$\cancel{-3}\quad -3$

$x=-3\pm 4$

$\boxed{x=1,-7}$

n) $-3\cdot 16=\cancel{-3}\cdot\dfrac{1}{\cancel{3}}(x-2)^2$

$\sqrt{-48}=\sqrt{(x-2)^2}$

$\pm i\sqrt{48}=x-2$

$4\ 12$

$4\ 3$

$\pm 4i\sqrt{3}=x\cancel{-2}$
$+2\qquad \cancel{+2}$

$\boxed{2\pm 4i\sqrt{3}=x}$

3.7 Completing the Square

FLASHBACK -- MULTIPLYING TRINOMIALS!!!

A Perfect Square Trinomial comes from multiplying two identical binomials together.

Ex: $(r+4)^2 = (r+4)(r+4) = r^2 + 8r + 16$

So if we had $(r+4)^2 = 25$ we could easily solve it by taking the Square Root of each side of the equation.

What if the trinomial that we have in an equation we are trying to solve is NOT a Perfect Square Trinomial? Can we turn it into one so that we can take the Square Root of each side to solve it?

To Solve a Quadratic Equation in *x* by Completing the Square:

1. Isolate the terms with variables on one side of the equation, and arrange them in descending order.
2. Divide both sides of the equation by the coefficient of x^2, if that coefficient is not 1.
3. Divide the coefficient of x by 2 then square your answer. Add the result to both sides of the equation. This is called completing the square.

   To complete the square for $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.

4. Factor the resulting perfect square trinomial and write it as the square of a binomial.

   $$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

5. Use the principle of square roots and solve for *x*.

Examples: Replace the blanks in each equation with constants to form a true equation.

a) $x^2 + 18x + \underline{81} = (x + \underline{9})^2$

9

b) $x^2 - 3x + \underline{9/4} = (x - \underline{3/2})^2$

$-\frac{3}{2}$

c) $x^2 + \frac{3}{5}x + \underline{9/100} = (x + \underline{3/10})^2$

$\frac{3}{10}$

Examples: Solve by completing the square. Show your work.

a) $x^2+4x-\cancel{12}=0$

$\quad\quad +\cancel{12}\ \ +12$

$x^2+4x+4=12$

$\quad\ 2 \quad\quad\quad +4$

$\sqrt{(x+2)^2}=\sqrt{16}$

$x+\cancel{2}=\pm 4$

$\ \ -\cancel{2}\quad -2$

$x=-2\pm 4$

$\boxed{x=2,-6}$

b) $x^2-6x-\cancel{6}=0$

$\quad\quad +\cancel{6}\ \ +6$

$x^2-6x+9=6$

$\quad -3 \quad\quad +9$

$\sqrt{(x-3)^2}=\sqrt{15}$

$x-\cancel{3}=\pm\sqrt{15}$

$\ \ +\cancel{3}\quad +3$

$\boxed{x=3\pm\sqrt{15}}$

c) $x^2+6x-\cancel{7}=0$

$\quad\quad +\cancel{7}\ \ +7$

$x^2+6x+9=7$

$\quad\ 3 \quad\quad +9$

$\sqrt{(x+3)^2}=\sqrt{16}$

$x+\cancel{3}=\pm 4$

$\ \ -\cancel{3}\quad -3$

$x=-3\pm 4$

$\boxed{x=1,-7}$

d) $x^2-16x+\cancel{59}=-7$

$\quad\quad -\cancel{59}\ \ -59$

$x^2-16x+64=-66$

$\quad -8 \quad\quad\quad +64$

$\sqrt{(x-8)^2}=\sqrt{-2}$

$x-\cancel{8}=\pm i\sqrt{2}$

$\ \ +\cancel{8}\quad +8$

$\boxed{x=8\pm i\sqrt{2}}$

e) $3x^2-6x-\cancel{9}=0$

$\quad\quad +\cancel{9}\ \ +9$

$3x^2-6x=9$

$3(x^2-2x+1)=9$

$\quad\quad -1 \quad\quad +3$

$\frac{\cancel{3}(x-1)^2}{\cancel{3}}=\frac{12}{3}$

$\sqrt{(x-1)^2}=\sqrt{4}$

$x-1=\pm 2$

$+1\quad +1$

$x=1\pm 2$

$\boxed{x=3,-1}$

f) $4x^2+8x+\cancel{3}=0$

$\quad\quad -\cancel{3}\ \ -3$

$4x^2+8x=-3$

$4(x^2+2x+1)=-3$

$\quad\quad 1 \quad\quad +4$

$\frac{\cancel{4}(x+1)^2}{\cancel{4}}=\frac{1}{4}$

$\sqrt{(x+1)^2}=\sqrt{\frac{1}{4}}$

$x+\cancel{1}=\pm\frac{1}{2}$

$\ \ -\cancel{1}\quad -1$

$x=-1\pm 1/2$

$\boxed{x=-1/2,-3/2}$

How did the Quadratic Formula come about? Well, it turns out that it can be derived by taking the Standard Form of a Quadratic Equation and then solve by Completing the Square.

Standard Form	⇨	$ax^2 + bx + c = 0$
Factor a from the "x" terms	⇒	$a\left(x^2 + \frac{bx}{a}\right) + c = 0$
Create a perfect square within the parentheses and add equivalent terms on both sides of the equation	⇒	$a\left(x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}\right) + c = \frac{b^2}{4a}$
Add - c to each side	⇒	$a\left(x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2\right) = \frac{b^2}{4a} - c$
Re-write RHS	⇒	$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$
Re-write the LHS	⇒	$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - \frac{4ac}{4a}$
Divide both sides by a	⇒	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
Take the square root	⇒	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$
Subtract $\frac{b}{2a}$ and re-write radical	⇒	$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
Re-write	⇒	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ⇦ **Quadratic Formula**

3.8 The Quadratic Formula and the Discriminant

The Quadratic Formula: A quadratic equation written in the form $ax^2 + bx + c = 0$, where $a \neq 0$, has the solutions: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Solving a Quadratic Equation Using the Quadratic Formula:

1. Write the equation in standard form: $ax^2 + bx + c = 0$.
2. Identify a, b, and c. Plug them into the equation. Be careful with parentheses.
3. Simplify. Be careful to follow order of operations and deal with negatives correctly.
4. Write your answers in simplified radical form. Use *i* when appropriate.

Examples: Solve each equation using the quadratic formula.

a) $x^2+4x+7=0$

$a=1,\ b=4,\ c=7$

$$x=\frac{-4\pm\sqrt{4^2-4(1)(7)}}{2(1)}$$

$$=\frac{-4\pm\sqrt{16-28}}{2}=\frac{-4\pm\sqrt{-12}}{2}$$

12: 4, 3; 4: 2, 2

$$=\frac{-4\pm 2i\sqrt{3}}{2}=\boxed{-2\pm i\sqrt{3}}$$

b) $3m^2+16m+5=0$

$a=3$
$b=16$
$c=5$

$$x=\frac{-16\pm\sqrt{16^2-4(3)(5)}}{2(3)}$$

$$=\frac{-16\pm\sqrt{256-60}}{6}=\frac{-16\pm\sqrt{196}}{6}$$

$$=\frac{-16\pm 14}{6}=\frac{-2}{6},\ \frac{-30}{6}=\boxed{-1/3,\ -5}$$

c) $2w^2-4w=3$

$a=2$
$b=-4$
$c=-3$

$2w^2-4w-3=0$

$$x=\frac{4\pm\sqrt{(-4)^2-4(2)(-3)}}{2(2)}$$

$$=\frac{4\pm\sqrt{16+24}}{4}=\frac{4\pm\sqrt{40}}{4}$$

40: 4, 10; 4: 2, 2; 10: 2, 5

$$=\frac{4\pm 2\sqrt{10}}{4}=\boxed{\frac{2\pm\sqrt{10}}{2}}$$

d) $-n^2+4n-4=0$

$a=-1$
$b=4$
$c=-4$

$$x=\frac{-4\pm\sqrt{4^2-4(-1)(-4)}}{2(-1)}$$

$$=\frac{-4\pm\sqrt{16-16}}{-2}=\frac{-4\pm\sqrt{0}}{-2}=\frac{-4}{-2}=\boxed{2}$$

e) $r^2+9=0$

$a=1$
$b=0$
$c=9$

$r^2+0r+9=0$

$$x=\frac{0\pm\sqrt{0^2-4(1)(9)}}{2(1)}$$

$$x=\frac{\pm\sqrt{-36}}{2}=\frac{\pm 6i}{2}$$

$$=\boxed{\pm 3i}$$

f) $6u^2-2u=0$

$a=6$
$b=-2$
$c=0$

$$x=\frac{2\pm\sqrt{(-2)^2-4(6)(0)}}{2(6)}$$

$$=\frac{2\pm\sqrt{4-0}}{12}=\frac{2\pm\sqrt{4}}{12}=\frac{2\pm 2}{12}$$

$$=\frac{4}{12},\ \frac{0}{12}=\boxed{\frac{1}{3},\ 0}$$

g) $z=-3z^2-3$

$a=-3$
$b=-1$
$c=-3$

$-3z^2-z-3=0$

$$x=\frac{1\pm\sqrt{(-1)^2-4(-3)(-3)}}{2(-3)}$$

$$=\frac{1\pm\sqrt{1-36}}{-6}=\frac{1\pm\sqrt{-35}}{-6}$$

$$=\boxed{\frac{-1\pm i\sqrt{35}}{6}}$$

h) $\frac{1}{4}y^2-y+\frac{1}{2}=0$

$a=\frac{1}{4}$
$b=-1$
$c=\frac{1}{2}$

$$x=\frac{1\pm\sqrt{(-1)^2-4(\frac{1}{4})(\frac{1}{2})}}{2(1/4)}$$

$$=\frac{1\pm\sqrt{1-1/2}}{1/2}$$

$$=\frac{1\pm i\sqrt{1/2}}{1/2}=\boxed{2\pm 2i\sqrt{1/2}}$$

Discriminant: The radicand of the quadratic equation, $\boxed{b^2 - 4ac.}$

The discriminant tells us about the number and types of solutions of a quadratic equation without actually solving it. It also tells us how many *x*-intercepts the graph of a function has.

Discriminant: $b^2 - 4ac$	Solutions of $ax^2 + bx + c = 0$
Positive	Two real solutions
Zero	One real solution
Negative	Two imaginary solutions

Examples: Find the discriminant of each quadratic equation and state the number and type (real or imaginary) of solutions.

a) $4x^2 - 20x + 25 = 0$

$a=4$ $(-20)^2-4(4)(25)$

$b=-20$ $400-400=\boxed{0}$

$c=25$

1 real solution

b) $x^2 + 2x + 4 = 0$

$a=1$ $2^2-4(1)(4)$

$b=2$ $4-16=\boxed{-12}$

$c=4$

2 imaginary solutions

c) $3x^2 + 5 = -7x$

$3x^2+7x+5=0$

$a=3$ $7^2-4(3)(5)$

$b=7$ $=49-60=\boxed{-11}$

$c=5$

2 imaginary solutions

d) $x^2 - 5x = 14$

$x^2-5x-14=0$

$a=1$ $(-5)^2-4(1)(-14)$

$b=-5$ $25+56=\boxed{81}$

$c=-14$

2 real solutions