Secondary Math 2H
Unit 3 Notes: Factoring and Solving Quadratics

3.1 Factoring out the Greatest Common Factor (GCF)

Factoring: The reverse of multiplying. It means figuring out what you would multiply together to get a polynomial, and writing the polynomial as the product of several factors (writing it as a multiplication problem).

Greatest Common Factor (GCF): The monomial with the largest possible coefficient and the variables with the largest possible exponents that divides evenly into every term of the polynomial.

Prime Polynomial: A polynomial that cannot be factored.

Factoring Out a Common Factor:

- Find the GCF.
- Use the distributive property in reverse to "factor out" the GCF:
  - Write the GCF outside a set of parentheses.
  - Inside the parentheses, write what you are left with when you *divide* the original terms by the GCF.
  - **Note:** If the GCF is the same as one of the terms of the polynomial, there will be a 1 left inside the parentheses.
- When the leading coefficient is negative, factor out a common factor with a negative coefficient.

Examples: Factor the following expressions.

a) x^2+3x

$x(x+3)$

b) $-2y+6$

$-2(y-3)$

c) $4n^2-20$

$4(n^2-5)$

d) $15d^2+20d^4$

$5d^2(3+4d^2)$

e) $2z^3+2z$

$2z(z^2+1)$

f) $-6h^2+3h$

$-3h(2h-1)$

g) $-20m^3+24m^2-32m$

$-4m(5m^2-6m+8)$

h) $-2a^2b^3c^4+8a^4b^8c^7-6a^3bc^5$

$-2a^2bc^4(b^2-4a^2b^7c^3+3ac)$

i) $p(q-6)+2(q-6)$

$(q-6)(p+2)$

Factoring by Grouping (4 Terms):

1. Factor out any common factors from all four terms first.
2. Look at the first two terms and the last two terms of the polynomial separately.
3. Factor out the GCF from the first two terms, write a plus sign (or a minus sign if the GCF on the third term is negative), then factor out the GCF from the last two terms.
4. You should have the same thing left in both sets of parentheses after you take out the GCFs. Factor out this common binomial factor from the two groups.

Examples: Factor the following expressions.

a) $x^3 - 4x^2 + 3x - 12$

$x^2(x-4) + 3(x-4)$

$(x^2+3)(x-4)$

b) $4y^3 + 2y^2 - 6y - 3$

$2y^2(2y+1) - 3(2y+1)$

$(2y^2-3)(2y+1)$

c) $20h^3 - 16h^2 - 5h + 4$

$4h^2(5h-4) - 1(5h-4)$

$(4h^2-1)(5h-4)$

d) $6q^3 + 2q^2r - 36q - 12r$

$2q^2(3q+r) - 12(3q+r)$

$(2q^2-12)(3q+r)$

3.2 Factoring Trinomials without a Leading Coefficient

Review Examples: Multiply the following.

a) $(x+3)(x+5)$

$x^2+5x+3x+15$

$x^2+8x+15$

b) $(n-7)(n-4)$

$n^2-4n-7n+28$

$n^2-11n+28$

c) $(t-2)(t+9)$

$t^2+9t-2t-18$

$t^2+7t-18$

d) Look at your answers. How do the numbers in your answer relate to the numbers in the factors?

The numbers in the factors multiply to the third term and add to the middle term.

Shortcut (only works if there's no number in front of the first term).

1. Find two numbers that multiply to c and add to b.
2. The factored form of x^2+bx+c is $(x+\mathbf{1st\ \#})(x+\mathbf{2nd\ \#})$.

Factor.

a) $x^2+11x+30$

$6 \cdot 5$

$\boxed{(x+6)(x+5)}$

$x^2+6x+5x+30$

$x(x+6)+5(x+6)$

$\boxed{(x+6)(x+5)}$

b) $m^2-8m+12$

$-6 \cdot -2$

$\boxed{(m-6)(m-2)}$

$m^2-6m-2m+12$

$m(m-6)-2(m-6)$

$\boxed{(m-6)(m-2)}$

c) $t^2+6t-40$

$10 \cdot -4$

$\boxed{(t+10)(t-4)}$

$t^2+10t-4t-40$

$t(t+10)-4(t+10)$

$\boxed{(t+10)(t-4)}$

d) q^2-16

$q^2+0q-16$

$-4 \cdot 4$

$\boxed{(q-4)(q+4)}$

$q^2-4q+4q-16$

$q(q-4)-4(q-4)$

$\boxed{(q-4)(q-4)}$

e) $w^2-18w+45$

$-15 \cdot -3$

$\boxed{(w-15)(w-3)}$

$w^2-15w-3w+45$

$w(w-15)-3(w-15)$

$\boxed{(w-15)(w-3)}$

f) $n^2-2n-35$

$-7 \cdot 5$

$\boxed{(n-7)(n+5)}$

$n^2-7n+5n-35$

$n(n-7)+5(n-7)$

$\boxed{(n-7)(n+5)}$

g) $-4x^2-16$

$\boxed{-4(x^2+4)}$

h) $a^2+7a-60$

$12 \cdot -5$

$\boxed{(a+12)(a-5)}$

$a^2+12a-5a-60$

$a(a+12)-5(a+12)$

$\boxed{(a+12)(a-5)}$

i) $5x^2+10xy+5y^2$

$5(x^2+2xy+y^2)$

$1 \cdot 1$

$\boxed{5(x+y)(x+y)}$

$5(x^2+xy+xy+y^2)$

$x(x+y)+y(x+y)$

$\boxed{5(x+y)(x+y)}$

g) h^3+h^2-12h

$h(h^2+h-12)$

$4 \cdot -3$

$\boxed{h(h+4)(h-3)}$

$h(h^2+4h-3h-12)$

$h(h+4)-3(h+4)$

$\boxed{h(h+4)(h-3)}$

h) $-5g^2+25g-30$

$-5(g^2-5g+6)$

$-2 \cdot -3$

$\boxed{-5(g-2)(g-3)}$

$-5(g^2-2g-3g+6)$

$g(g-2)-3(g-2)$

$\boxed{-5(g-2)(g-3)}$

i) $5x^2-20$

$5(x^2-4)$

$2 \cdot -2$

$\boxed{5(x+2)(x-2)}$

$5(x^2+2x-2x-4)$

$x(x+2)-2(x+2)$

$\boxed{5(x+2)(x-2)}$

3.3 Factoring a Trinomial with a Leading Coefficient

Review Examples: Factor the following.

1. $x^2+3x-28$

$7\cdot -4$

$x^2+7x-4x-28$

$x(x+7)-4(x+7) \rightarrow \boxed{(x+7)(x-4)}$

2. $x^2-9x+18$

$-6\cdot -3$

$x^2-6x-3x+18$

$x(x-6)-3(x-6) \rightarrow \boxed{(x-6)(x-3)}$

3. Look at your factors. How do the factors relate to the numbers in the original problem?

~~Guess and Check~~ Trinomials of the Form ax^2+bx+c

1. **List the factors of the first term.**
2. **List the factors of the last term.**
3. **Choose the factors where the multiples add to the middle term.**
4. **Check by multiplying**

a) $5x^2+7x+2$

b) $3y^2-8y+4$

c) $6c^2+c-2$

d) $9m^2+9m+2$

e) $2a^2-11a+12$

f) $5d^2+18d-8$

g) $6r^2-8r-8$

h) $15x^2-19x-10$

i) $8g^2-24g+18$

Factoring a Trinomial of the Form $ax^2 + bx + c$ by Grouping:

a) Always check for a GCF first! If there is a GCF, factor it out.
b) Multiply $a \cdot c$.
c) Find two numbers that multiply to your answer $(a \cdot c)$ and add to b.
d) Rewrite the middle term bx as **1st** $\# \cdot x +$ **2nd** $\# \cdot x$
e) Factor the resulting polynomial by grouping.
f) If there are no numbers that multiply to $a \cdot c$ and add to b, the polynomial is prime.

Examples: Factor the following polynomials.

f) $9h^2 + 9h + 2$ 18 $6 \cdot 3$

$9h^2 + 6h + 3h + 2$

$3h(3h+2) + 1(3h+2)$

$\boxed{(3h+2)(3h+1)}$

g) $2z^2 - 11z + 12$ 24 $-8 \cdot -3$

$2z^2 - 8z - 3z + 12$

$2z(z-4) - 3(z-4)$

$\boxed{(z-4)(2z-3)}$

h) $3r^2 - 16r - 12$ -36 $-18 \cdot 2$

$3r^2 - 18r + 2r - 12$

$3r(r-6) + 2(r-6)$

$\boxed{(r-6)(3r+2)}$

i) $3x^2 + 19x + 15$ 45 $3 \cdot 3 \cdot 5$

$\boxed{\text{prime}}$

j) $10m^2 + 13m - 3$ -30 $15 \cdot -2$

$10m^2 + 15m - 2m - 3$

$5m(2m+3) - 1(2m+3)$

$\boxed{(2m+3)(5m-1)}$

k) $4p^2 - 20p + 21$ 84 $2 \cdot 2 \cdot 3 \cdot 7$ $-14 \cdot -6$

$4p^2 - 14p - 6p + 21$

$2p(p-7) - 3(p-7)$

$\boxed{(p-7)(2p-3)}$

l) $4n^2 - 20n + 25$ 100 $-10 \cdot -10$

$4n^2 - 10n - 10n + 25$

$2n(2n-5) - 5(2n-5)$

$\boxed{(2n-5)(2n-5)}$

m) $12y^2 + 30y - 72$

$6(2y^2 + 5y - 12)$ -24 $8 \cdot -3$

$6(2y^2 + 8y - 3y - 12)$

$2y(y+4) - 3(y+4)$

$\boxed{6(y+4)(2y-3)}$

n) $8k^4 + 42k^3 - 36k^2$ $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

$2k^2(4k^2 + 21k - 18)$ -72 $24 \cdot -3$

$2k^2(4k^2 + 24k - 3k - 18)$

$4k(k+6) - 3(k+6)$

$\boxed{2k^2(k+6)(4k-3)}$

o) $4x^2 - 2xy - 12y^2$

$2(2x^2 - xy - 6y^2)$ -12 $-4 \cdot 3$

$2(2x^2 - 4xy + 3xy - 6y^2)$

$2x(x-2y) + 3y(x-2y)$

$\boxed{2(x-2y)(2x+3y)}$

p) $8m^2 + 18m + 4$

$2(4m^2 + 9m + 2)$ 8 $8 \cdot 1$

$2(4m^2 + 8m + m + 2)$

$4m(m+2) + 1(m+2)$

$\boxed{2(m+2)(4m+1)}$

q) $9x^3 - 20x^2 + 4x$

$x(9x^2 - 20x + 4)$ 36 $-18 \cdot -2$

$x(9x^2 - 18x - 2x + 4)$

$9x(x-2) - 2(x-2)$

$\boxed{x(x-2)(9x-2)}$

3.4 Perfect Square Trinomials and Differences of Squares

Perfect Square Trinomials

Review Examples: Multiply the following:

a) $(a+6)(a+6)$

$a^2+6a+6a+36$

$a^2+12a+36$

b) $(2m-3)(2m-3)$

$4m^2-6m-6m+9$

$4m^2-12m+9$

c) $(4-k)(4-k)$

$16-4k-4k+k^2$

$16-8k+k^2$

- **To Recognize a Perfect-Square Trinomial**
  - Two terms must be squares, A^2 and B^2.
  - The remaining term must be $2AB$ or its opposite, $-2AB$.

- **To Factor a Perfect-Square Trinomial**
  - $A^2+2AB+B^2=(A+B)^2$
  - $A^2-2AB+B^2=(A-B)^2$

Examples: Factor the following polynomials.

a) $x^2+10x+25$ $5\cdot5$

$(x+5)^2$

$x^2+5x+5x+25$

$x(x+5)+5(x+5)$

$(x+5)(x+5)$

$(x+5)^2$

b)) $x^2-14x+49$ $-7\cdot-7$

$(x-7)^2$

$x^2-7x-7x+49$

$x(x-7)-7(x-7)$

$(x-7)(x-7)$

$(x-7)^2$

c) $4x^2+12x+9$ 36 $6\cdot6$

$(2x+3)^2$

$4x^2+6x+6x+9$

$2x(2x+3)+3(2x+3)$

$(2x+3)(2x+3)$

$(2x+3)^2$

d) $4y^2+16y+16$

$(2y+4)^2$ or $4(y^2+4y+4)$ $2\cdot2$

$4(y+2)^2$

$4(y^2+2y+2y+4)$

$\cancel{y(y+2)}+2(y+2)$

$\boxed{4(y+2)^2}$

e) $9r^2+48r+64$ 576 $24\cdot24$

$(3r+8)^2$

$9r^2+24r+24r+64$

$3r(3r+8)+8(3r+8)$

$(3r+8)(3r+8)$

$(3r+8)^2$

f) $49y^2-84y+36$ 1764 $-42\cdot-42$

$(7y-6)^2$

$49y^2-42y-42y+36$

$7y(7y-6)-6(7y-6)$

$(7y-6)(7y-6)$

$\boxed{(7y-6)^2}$

g) $81-90a+25a^2$ 2025 $-45\cdot-45$

$(9-5a)^2$

$81-45a-45a+25a^2$

$9(9-5a)-5a(9-5a)$

$(9-5a)(9-5a)$

$(9-5a)^2$

Difference of Squares

Review Examples: Multiply the following:

a) $(a+4)(a-4)$

$a^2 - \cancel{4a} + \cancel{4a} - 16$

$a^2 - 16$

b) $(3-k)(3+k)$

$9 + \cancel{3k} - \cancel{3k} - k^2$

$9 - k^2$

c) $(2m+7)(2m-7)$

$4m^2 - \cancel{14m} + 14m - 49$

$4m^2 - 49$

Factoring a Difference of Squares:

- A polynomial of the form $A^2 - B^2$ is called a ***difference of squares.***
- Differences of squares always factor as follows: $A^2 - B^2 = (A+B)(A-B)$

★ This only works if ***both terms are perfect squares*** **and** ***you are subtracting.*** Don't forget to check for a GCF first!

Examples: Factor the following polynomials.

a) $x^2 - 25$

$(x+5)(x-5)$

$x^2 + 0x - 25$ $\quad -25$, $-5 \cdot 5$

$x^2 - 5x + 5x - 25$

$x(x-5) + 5(x-5)$

$(x-5)(x+5)$

b) $m^2 - 81$

$(m+9)(m-9)$

$m^2 + 0m - 81$ $\quad -81$, $-9 \cdot 9$

$m^2 - 9m + 9m - 81$

$m(m-9) + 9(m-9)$

$(m-9)(m+9)$

c) $w^2 + 36$

prime

$w^2 + 0w + 36$ $\quad 36$

prime

d) $49 - n^2$

$(7+n)(7-n)$

$49 + 0n - n^2$ $\quad -49$, $-7 \cdot 7$

$49 - 7n + 7n - n^2$

$7(7-n) + n(7-n)$

$(7-n)(7+n)$

e) $4t^2 - 1$

$(2t+1)(2t-1)$

$4t^2 + 0t - 1$ $\quad -4$, $-2 \cdot 2$

$4t^2 - 2t + 2t - 1$

$2t(2t-1) + 1(2t-1)$

$(2t-1)(2t+1)$

f) $z^4 - 64$

$(z^2+8)(z^2-8)$

$z^4 + 0z^2 - 64$ $\quad -64$, $-8 \cdot 8$

$z^4 - 8z^2 + 8z^2 - 64$

$z^2(z^2-8) + 8(z^2-8)$

$(z^2-8)(z^2+8)$

g) $64y^4 - 81x^2$

$(8y^2+9)(8y^2-9)$

h) $144k^2 + 25$

prime

i) $2a^2 - 242$

$2(a^2 - 121)$

$2(a+11)(a-11)$