10.2 Permutations and Combinations

Independent and Dependent Events

VOCABULARY -

- ***Outcome:*** The result of a single trial, experiment, or decision.
  - The possible outcomes of a coin flip would be heads or tails.
  - The possible outcomes of rolling a single die would be 1, 2, 3, 4, 5, or 6.

- ***Sample Space:*** The set of all possible outcomes.
  - The sample space for a coin flip is $\{H, T\}$.
  - The sample space for two coin flips would be $\{HH, HT, TH, TT\}$

- ***Event:*** One or more possible outcomes of a trial.
  - A coin flip coming up heads
  - Rolling either a 2 or a 6 on a die
  - Choosing a face card from a deck of cards

- ***Independent Events:*** The outcome of one event *does not* affect the outcome of another event.
  - Two coin flips are independent because the outcome of the first coin flip does not affect the outcome of the second flip.

- ***Dependent Events:*** The outcome of one event *does* affect the outcome of another event.
  - Choosing a piece of candy from a jar and then choosing a second piece without replacing the first are dependent events because taking the first piece affects what is available to be taken next.

Examples: Decide whether the events are *dependent* or *independent.*

1. Choosing an ice cream flavor and then choosing a topping for the ice cream.

   Independent

2. Choosing one book on which to write an essay on and then a different book on which to give a presentation.

   Dependent

3. Awarding 1st, 2nd, and 3rd prize to entries in an art contest.

   Dependent

4. Selecting a fiction book and a nonfiction book from the library.

   Independent

Permutation and Combinations

- ***Permutation***: An arrangement of a group of objects, where order matters.
  Ex: Selecting a president, vice president, and secretary/lock combination/batting order

Permutation Formula (no repetition allowed)

$_nP_r = P(n,r) = \boxed{\dfrac{n!}{(n-r)!}}$ where n is the number of things you choose from and you choose r of them

- ***Combination***: An arrangement or selection of objects in which order is not important.
  Ex: Choosing Pizza toppings/selecting people to a committee

Combination Formula ~~(no repetition allowed)~~

$_nC_r = C(n,r) = \binom{n}{r} = \boxed{\dfrac{n!}{r!(n-r)!}}$ where n is the number of things you choose from and you choose r of them

order   no order

Examples: Determine whether each situation involves a permutation or a combination.

1. In how many different orders can a person read 5 separate magazines?

   order –   permutation

2. You have just purchased 15 new songs and want to add them to a playlist. You don't want to remove any of the songs that are already on your playlist and there is only room for 5 more tracks. How many ways can you add 5 different tracks to the playlist?

   order   Permutation

3. You have 7 new movies on Netflix that you are dying to watch. You only have time to watch 3 this weekend. How many distinct ways can you watch the movies this weekend?

   Order   Permutation

4. Honors English students are required to read 8 books from a list of 25. How many combinations could a student select?

   no order   Combination

5. You are Mr. Manager of a frozen banana stand. You need 5 employees and have 20 qualified applicants. How many ways can you staff the store?

   no order   Combination

6. Four members from a group of 18 on the board of directors at the Fa La La School of Arts will be selected to go to a convention (all expenses paid) in Hawaii. How many different groups of 4 are there?

$_nP_r$ or $P(n,r) = \frac{n!}{(n-r)!}$ order

$C(n,r)$ or $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ no order

Examples: Evaluate each expression

1. $_6P_2$ $\frac{6!}{(6-2)!} = \frac{6\cdot5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{4}\cancel{3}\cancel{2}\cancel{1}} = \boxed{30}$

2. $P(10,5) = \frac{10!}{(10-5!)} = \frac{10\cdot9\cdot8\cdot7\cdot6\cdot\cancel{5!}\cdots}{\cancel{5!}}$ $\boxed{30,240}$

3. $_9P_6$ $\frac{9!}{(9-6)!} = \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot\cancel{3!}}{\cancel{3!}}$ $3!$ $\boxed{60,480}$

calc. 4. $P(25,20)$ $\frac{25!}{(25-20)!} = \frac{25\cdot24\cdot23\cdot22\cdot21\cdot20!\cdots}{5!}$ $\boxed{1.292600837E23}$

5. $_{15}C_4 = \frac{15!}{4!(15-4)!} = \frac{15\cdot\cancel{14}\cdot13\cdot\cancel{12}\cdot\cancel{11}\cdots}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1\ \cancel{11}\cdots}$ $11!$ $\boxed{1365}$

6. $_9C_6$ $\frac{9!}{6!(9-6)!} = \frac{\overset{3}{\cancel{9}}\cdot\overset{4}{\cancel{8}}\cdot7\cdot\cancel{6!}}{\cancel{6!}\ \cancel{3}\cdot\cancel{2}\cdot1} = \boxed{84}$

7. $C(12,4)$ $\frac{12!}{4!(12-4)!} = \frac{\cancel{12}\cdot11\cdot\overset{5}{\cancel{10}}\cdot9\cdot\cancel{8!}}{\cancel{4}\cancel{3}\cdot\cancel{2}\cdot1\ \cancel{8!}}$ $\boxed{495}$

8. $\binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{9\cdot\overset{4}{\cancel{8}}\cdot\cancel{7!}}{\cancel{2}\cdot1\ \cancel{7!}} = \boxed{36}$

Examples: Determine whether each situation involves a *permutation* (order) or a *combination* (no order). Then find the number of possibilities.

1. Twelve (n) skiers are competing in the final round of the Olympic freestyle skiing aerial competition. In how many ways can 3 (r) of the skiers finish first, second, and third to win the gold, silver, and bronze medals?

$\boxed{\text{Permutation}}$ $_{12}P_3 = \frac{12!}{(12-3)!} = \frac{12\cdot11\cdot10\cdot\cancel{9!}}{\cancel{9!}} = \boxed{1320}$ $12\cdot11\cdot10$

2. A pizza shop offers twelve (n) different toppings. How many different three-topping pizzas can be formed with the twelve toppings?

no order $\boxed{\text{Combination}}$ $_{12}C_3 = \frac{12!}{3!(12-3)!} = \frac{\overset{2}{\cancel{12}}\cdot11\cdot10\cancel{9!}}{\cancel{3}\cdot\cancel{2}\cdot1\ \cancel{9!}} = \boxed{220}$

3. Your English teacher has asked you to select 3 novels from a list of 10 to read as an independent project. In how many ways can you choose which books to read?

$\boxed{\text{Combination}}$ $_{10}C_3 = \frac{10!}{3!(10-3)!} = \frac{10\cdot\overset{3}{\cancel{9}}\cdot\overset{4}{\cancel{8}}\cdot\cancel{7!}}{\cancel{3}\cdot\cancel{2}\cdot1\ \cancel{7!}} = \boxed{120}$

4. The school yearbook has an editor-in-chief and an assistant editor-in-chief. The staff of the yearbook has 15 students. How many different ways can students be chosen for these 2 positions?

order $\boxed{\text{Permutation}}$ $_{15}P_2 = \frac{15!}{13!} = \frac{15\cdot14\cdot\cancel{13!}}{\cancel{13!}} = \boxed{210}$

It is important to note that when you use the permutation or combination formulas above, repetition is not allowed. In other words, you can't have the same person win first and second place.

Another Case to Consider

Example: How many different ways can the letters HTAM be arranged to create four-letter "words"?

Solution: This is an example of a permutation because the order of the letters would produce a different "word" or outcome. So, we use the permutation formula:

$${}_4P_4 = P(4,4) = \frac{4!}{(4-4)!} = \frac{4!}{0!} = \frac{4\cdot3\cdot2\cdot1}{1} = 24$$

4 3 2 1

But, what if some of the letters repeated? For example, how many ways can the letters in CLASSES be rearranged to create 7 letter "words"? Since the letter S repeats 3 times, some of the permutations will be the same so we will have to eliminate them. Here is how we do it. There are 7 letters to choose from and we are choosing 7 of them, so we would have the following:

$${}_7P_7 = P(7,7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1} = 5040$$ Hold on – this is not our answer yet. We have to divide out our duplicate letters. As we mentioned earlier, the letter S repeats 3 times so we divide our answer by 3!:

$$\frac{5040}{3!} = 840$$

Example 7: How many ways can the letters in MISSISSIPPI be arranged to create 11-letter "words"?

I repeats 9 times
S repeats 4 times
P repeats 2 times

$$P(11,11) = \frac{11!}{(11-11)!} = \frac{11!}{0!} = 11!$$

Divide out repeats →

$$\frac{11!}{4!\cdot4!\,2!} = \frac{11\cdot10\cdot9\cdot\cancel{8}\cdot7\cdot\cancel{6}\cdot5\cdot\cancel{4!}}{\cancel{4!}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1\cdot\cancel{2}\cdot1}$$

$$\boxed{34650}$$